On the problem of LC-circuits eigenvalues multiplicity

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Abstract

Weinstein method of intermediate problems has been modified for a system with concentrated parameters – electrical circuits. Basis (initial) problem for this case of the intermediate problems method is defined. A relationship between eigenvalues (proper frequencies) of impedances of separate branches of the circuit and loop impedances are established. A simple technique of separating the roots of characteristic polynomials is elaborated. Finite steps recurrent process of intermediate problems of eigen values is determined. The latter leads to the important result conserning LC-circuit eigenvalues multiplicity, in the synthesis of circuits with a given range of eigenfrequencies by simply choosing the required number of elements (impedances) of the same kind in a primitive circuit.

Keywords: eigen values spectrum, electrical circuit, intermediate problems,orthogonal circuits, pure-loop, pure-node,roots multiplicity, Weinstein's Function

Introduction

To begin with, we have to state shortly basic conceptions of tensorial theory of electrical circuits (Kron, 1959; Happ, 1973) and some our previous results connected with the theory (Mylnikov & Prangishvili, 2002; Milnikov, 2013; Mylnikov, 2008). Four types of circuits (introduced by Kron (Kron, 1959]) are used in the present paper: pure-loop, pure-node, orthogonal and primitive. The first one is a circuit which consists only of loops, on the contrary, a pure-node circuit consists only of node pairs, orthogonal circuits are ordinary circuits with both loops and node pairs and primitive circuit is a circuit consisting of disconnected branches (Kron, 1959). A pure-loop circuit can be easily obtained from an ordinary, i.e. orthogonal circuit: if we have a k loop circuit, then we should shortcircuit n-k= m-1node pairs . However, in the case of node analysis leading to pure-node circuits, we are to do a dual operation: to open k loops.

In different works (Mylnikov & Prangishvili, 2002; Milnikov, 2013; Mylnikov, 2008) it has been shown that to each circuit, one can assign two pairs of conjugate linear vector spaces HL^n, HL_n and CL^n, CL_n one of which has a homological origin, while the other one—cohomological. Four spaces generate two pairs of conjugate variables e, i and E, I. Also invariance of input (homological) and output (cohomological) powers was proved. The latter allowed us to substantiate tensorial model of multiloop electrical circuit. From this point of view, one can consider the mesh current method as the tensor form of Ohm's law written for k-dimensional homological spaces HL^n and HL_n , while the node voltage method is the tensor form of Ohm's law written for m − 1=n-k-dimensional cohomological spaces CL^{n-k} and CL_{n-k} . The kinetic (magnetic) energy of the circuit is a bilinear form to which there corresponds a twice covariant inductance (mass) tensor. The potential (electric) energy of the circuit is a bilinear form to which there corresponds a twice contravariant capacitance (elasticity) tensor (Mylnikov & Prangishvili, 2002; Milnikov, 2013; Mylnikov, 2008) .Another result important for the following is that to a given primitive circuit, one can assign the group GC of transformations C, which completely describes all possible kinds of pure-loop circuits can be obtained from the initial primitive circuit.

Hereafter we use notation for eigenvalues λ, which is equal to the second power of angular frequency $\lambda = \omega^2$.

Problem Formulation

Weinstein's method of intermediate problems was developed for infinite-dimensional problems, for which it proved to be sufficiently effective, especially for problems connected with oscillations of membranes of various configurations (Mylnikov, 2008). However, the part of the method that was developed for finite-dimensional problems had no practical importance.The reason is obvious: the application of Weistein's function and especially of Aronszajn's lemma require the resolvent be calculated for each tested value, which is absolutely impossible to do in case of large (many-loop) circuits. Another point, which is probably the main one, in the finitedimensional case is not clear how to use Aronszajn's lemma in general, since the method of intermediate problems does not give any clues as to how one can construct the so-called basic problem. All the mentioned prob-

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lems associated with Weinstein's function and Aronszajn's lemma have become solvable in next to no time and lead to new significant results if this method is modified so as to conform to the notions of primitive and pure-loop (pure-node) circuits.

Finally, we have to state that the problems connected with synthesis of multiloop LC- circuits with predefined resonance frequencies (eigenvalues) is one of the most demanded in developing of different modern digital and analog communication devices. With this in view, various methods are used, but the majority of them are complicated and time consuming. The objective of the present paper is a presentation of new approach to the problem of synthesis of multiloop LC- circuits with predefined resonance frequencies based on Midified Intermediate Problems method (Milnikov, 2013; Mylnikov, 2008) and G. Kron's conceptions (Kron, 1959).

Problem Solution

I.1 Determination of the Basic (Initial) Problem for the Intermediate Problems Method for Electrical Circuits (Milnikov, 2013)

Let us transform the initial K-loop circuit to a pure-loop circuit by shorting n- k node pairs. To this circuit there corresponds the ndimensional operator $Z^{(n)}$. Moreover, we have a primitive circuit, to which there corresponds also an n-dimensional operator Z.D, which matrix is diagonal with the diagonal $\lambda l_{ii} - 1/c^{ii}$ (i=1,2,…,n).

The matrices $Z^{(n)}$ and Z_D are related through\

$$Z^{(n)}(\omega) = C^T Z_D(\omega) C \tag{1}$$

where C is an n × n matrix of transformation of the initial primitive circuit to the connected pure-loop one.

The inverse matrix to (1) is the resolvent R_λ^o for a primitive circuit with diagonal elements inversed to diagonal elemnts Z_D : $1/(\lambda l_{ii} - 1/c^{ii})$.

Using (1) it is easy to obtain the resolvent for $Z^{(n)}$

$$R_\lambda^{(n)} = C^{-1} R_\lambda^{(0)} (C^{-1})^T \tag{2}$$

We would like to emphasize the fact that (2) is in fact the resolvent obtained in a general form so that we need not to calculate it anew (i.e. to transform the matrix) for each tested λ .

The eigenvalues of the diagonal operator Z_D are obviously equal to $\lambda_i = 1/(l_i c^i)$ (i=1,2,…n). Among them there may be multiple eigenvalues too, which from the engineering standpoint means that among the elements used to con-struct a k-loop circuit there are groups of elements having the sameimpendance values and the quantities of these groups are equal to the eigen values corresponding multiplicities.

Proposition 1. All pure-loop circuits contained in the group G_C of the initial primitive circuit possess pairwise equal eigenvalues equal in their turn to the eigenvalues of the primitive circuit.

Proof. From (2) it follows that

$$\det(R_\lambda^{(n)}) = \det(C^{-1}) \det(R_\lambda^{(o)}) \det(C^{-1})^T$$

But det C^T and detC are constant values and therefore the respectivedeterminants are equal to zero only for equal λ .

Analogously, for two arbitrary pure-loop circuits, each of which isobtained by means of a nonsingular transformation C from a givenprimitive circuit, we can write

$$\det(Z_j^n) = \det(C_{ji}^T (Z_i^n) C_{ij}) \tag{3}$$

$Z_j^{(n)}$ is the impendance tensor of the i-th pure-loop circuit; isthe impendance tensor of the j-th pure-loop circuit $Z_j^{(n)}$; C_{jiis} the tensorof transformation of the basis of the i-th pure-loop circuit to that ofthe j-th circuit.

From (3) it follows that the equality

$$\det(Z_j^{(n)}) = \det(Z_i^{(n)}) = \det(Z_0^{(n)}) = 0$$

is again fulfilled for equal λ , Q.E.D.

Thus, we have obtained two important results: – the resolvent of the operator $R_\lambda^{(n)}$ of a pure-loop circuit can be obtained directly from (2) without transforming the matrix $Z^{(n)}$ and the eigenvalues of the initial primitive system are equal to the eigenvalues of a pure-loop circuit or, in other words, the eigenfreqiencies of individual elements, by which the circuit is constructed, are equal to the eigenfreqiencies of the constructed circuit where n-k node pairs are shorted.

The above reasoning has been carried out using the terms of the method of loop currents. The same can also be done in terms of node voltages. We are omitting the consideration of the case due to lack of space, but only note that in this case the admittance tensor Y^n should be used.

Propositions 1 implies that in the method of intermediate problems we should consider as basic problems either a pure-loop circuit or a pure-node circuit because the resolvents of these problems are easily defined in a general form, and the eigenvalues are likewise easily calculated. As the basis operator we should consider the impedancetensor $Z^{(n)}$ of a pure-loop circuit (the admittance Y^n tensor in the case of a pure-node circuit).Consecutive imposing of constraints on the pure-loop circuit generates sequence ofrespective intermediate operators

$$Z^{(n-1)}, Z^{(n-2)}, ..., Z^{(k)}$$

Eigen values of the latter operator represent our original problem.

I.2 The Weinstein function for oscillatory circuits (Mylnikov, 2008; Milnikov & Duisheev, 2014)

We proceed from the fact that the operator can beobtained from the operator Z(n)by opening successively n-k short-circuited node pairs of a pure-loop circuit, which is equivalent to imposingn-k constraints.

If one number of all n loops so that fictious n-k loops would get thelast n-k numbers, then the opening of the j-th

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loop obviously leads tothe constraint equation

$$i_j = 0 \tag{4}$$

To this equation there corresponds the constraint vector pj=(0; 0; … ; 1; …, 0), where 1 is in the j-th position. Thus, the k-loop circuit is obtained from the corresponding pure-loop circuit by imposing successively (or simultaneously) n-k constraints to which there correspond n-k mutually orthogonal, unit basis constraint vectors p.

From the geometric standpoint, the process of imposing r constraints corresponds to the transformation of the operator to its part , which is defined on the subspace of the space . Now we can obtain the concrete representation of the part of the operator .

Proposition 2. The operator $Z^{(n-r)}$ which is a part of the operator $Z^{(n)}$ and defined on the subspace LH^{n-r} is represented in the coordinate from as a principal submatrix[1] of order n-r of the matrix $Z^{(n)}$.

We are omitting the proof of the proposition 2 which can be found in (Mylnikov, 2008).

Thus, when constraints of type (4) are successively imposed on a pure-loop circuit, we obtain a number of intermediate problems on eigenvalues for a chain of operators

$$Z^{(n-1)}, Z^{(n-2)}, ..., Z^{(k)}$$

each of which is in coordinate terms a principal submatrix (of order smaller by one) of the preceding operator:

$$Z^{(n-1)}, Z^{(n-2)}, ..., Z^{(n-i)}, Z^{(n-(i+1))}, ..., Z^{(k)}$$

Similar to (1), one can write down a transformation for pure-node circuits

$$Y^{(n)}(\omega) = A^T Y_D(\omega) A \tag{1´}$$

where - covariant tensor A, which connects the conductance tensor of a primitive circuit with the tensor of an orthogonal pure-node circuit

It has been shown that the tensors A and C are related by

$$A^T = C^{-1} \text{ [A. Mylnikov 2008].} \tag{5}$$

If the matrices A and C are divided into blocks in accordance with the division of circuit variables into k loop (contravariant) and n-k node variables (covariant), then it turns out that the matrices of the tensors C and A have the following block structures:

$$C = |C_k C_{n-k}| \quad A = |A_k A_{n-k}|$$

where C_k and A_{n-k} coincide with the loop and structural matrices of thecircuit.

Proposition 3. The matrices $Z^{(n)}(\omega)$ and $Y^{(n)}(\omega)$ are reciprocal, i.e. $Y^{(n)}(\omega)$ is theresolventfor $Z^{(n)}(\omega)$, and vice versa.

Indeed, taking into account $Z_D(\lambda) = (Y(\lambda))^{-1}$ and also equality (1) one can write

$$(Y^{(n)}(\lambda))^{-1} = (A^T Y_D(\lambda) A)^{-1} =$$
$$= A^{-1} Y_D^{-1}(\lambda)(A^T)^{-1} = C^T Z_D(\lambda) C = Z^{(n)}(\lambda).$$

Q.E.D.

Now one can determine the shape of Weinstein function for finite dimensional discrete system – k - loop circuits. If as a basis operator we take $Z^{(n)}(\omega)$, then by Proposition 2 its resolvent is $Y^{(n)}(\omega)$ and for an arbitrary LC-circuit we can write the Weinstein function as follows:

$$W(\lambda) = \left| Y^{(n)}(\lambda) p_i, p_j \right| \text{ (i,j=n.n-1,…,k+1),} \tag{6}$$

where pi – a vector of imposed i-thconstraint.Performing all multiplication operations in (6) we obtain the determinant of the matrix of (n-k) order, lying at the intersection of the last n-k rows and columns of the matrix $Y^{(n)}$. This gives rise to

Proposition 4. The Weinstein function for the LC-circuit described by the loop matrix Z(k) is the determinant of a lower right submatrix of order (n-k) of the resolvent Y(n).The dual statement is also valid.

Proposition 4´. The Weinstein function for the LC-circuit described by the node matrix $Y^{(n-k)}$ is the determinant of an upper submatrix of order k of the resolvent $Z^{(n)}$.

One can easily establish a relation between these submatrices.

Lemma 1. A lower right submatrix of order (n-k) of the resolvent $Y^{(n)}$ is a node conductance matrix $Y^{(n-k)}$.

Indeed, rewriting (1´) in the block form and performing multiplication, we have

$$Y^{(n)}(\lambda) = \begin{vmatrix} A_k^T \\ A_{n-k}^T \end{vmatrix} \cdot |Y_D(\lambda)| \cdot |A_k A_{n-k}| =$$
$$\begin{vmatrix} A_k^T Y_D(\lambda) A_k & A_k^T Y_D(\lambda) A_{n-k} \\ A_{n-k}^T Y_D(\lambda) A_k & A_{n-k}^T Y_D(\lambda) A_{n-k} \end{vmatrix}$$

The block located in the right lower corner is the node conductance matrix Y(n-k) by virtue of the fact that Acoincides with thestructural matrix of the circuit. Q.E.D.

The dual statement is proved analogously.

Lemma 1´. An upper left upper submatrix of order k of the resolvent $Z^{(n)}$ is a loop impendence matrix $Z^{(k)}(\lambda)$.

Propositions 4 and 4´, Lemmas 1 and 1´ immediately imply

Proposition 5. The determinant of the conductance node matrix of an arbitrary k-loop LC-circuit is the Weinstein

1 A matrix located at the intersection of the first r rows and r columns is called a principal submatrix of order r of an arbitrary square matrix A of order n.

function for the loop impendence matrix of this circuit, and vice versa.

The latter proposition establishes a deep relationship of the classical loop current and node potential methods with the operator methods of many-dimensional geometry. Thus, the determinant

$$A_{n-k}^T Y_D(\lambda) A_{n-k}$$

is, on the one hand, the Weistein function obtained by imposing n-k constraints on the resolvent $Y^{(n)}(\lambda)$ of the operator $Z^{(n)}(\lambda)$ and, on the other hand, its matrix is the node conductance matrix of the considered circuit. Conversely, if the circuit is considered in terms of node analysis, then the Weinsten function $C_k^T Z_D(\lambda) C_k$ obtained byimposing k constraints on the resolvent $Z^{(n)}(\lambda)$ of the operator $Y^{(n)}(\lambda)$ is the loop resistance matrix of the analyzed circuit.

I.3 Roots Separation and Multiplicity (Mylnikov, 2008)

It is obvious that when i constraints are imposed on a pure loop circuit we obtain new eigenvalues, which allows us to introduce

Definition 1. Eigenvalues of the base oscillatory system are called eigenvalues of zeroth order (they correspond to the operator $Z^{(n)}(\lambda)$, while eigenvalues obtained by imposing i constraints on a pure loop circuit (they correspond to the operator $Z^{(n-i)}(\lambda)$) are called eigenvalues of i-th order.

It is obvious that in the light of the introduced terminology the eigenvalues of the considered k-loop circuit are eigenvalues of n-k-th order (they correspond to the operator $Z^k(\lambda)$. Thus, to each k-loop circuit there correspond n-k series of eigenvalues

$$\lambda_1^{(n)},...,\lambda_n^{(n)};\lambda_1^{(n-1)},...,\lambda_{n-1}^{(n-1)};...\lambda_1^{(n-k)},...,\lambda_k^{(n-k)}$$

(the last n-k-th series consisting of eigenvalues in the usual sense)

Note that eigenvalues of zeroth order $\lambda_1^{(n)},...,\lambda_n^{(n)}$ are in fact eigenvalues of individual impedances that make up a k-loop circuit and therefore in the general case the notion of an eigenvalue of zero order does not coincide with the notion of a partial frequency. However eigenvaleus of individual impedances are partial frequencies for a pure loop circuit.

Also note that the presence of multiple eigenvalues of zeroth order testifies to the existence of groups of impedances of the same kind in a primitive circuit, the number of elements of each group being equal to the corresponding multiplicity.

Using Rayleigh' theorem one can prove

Proposition 6. The eigenvalues of the operator $Z^{(i-1)}(\lambda)$ separate the eigenvalues of the operator $Z^{(i)}(\lambda)$.

Indeed, the operator is obtained from the operator by imposing one constraint (by opening one fictious loop). By Rayleigh's theorem this means that the eigenvalue of both operators satisfy inequalities

$$\lambda_j^{(n-i)} \le \lambda_j^{(n-i+1)} \le \lambda_{j+1}^{(n-i)} \qquad (7)$$

Q.E.D.

From the Proposition 6 it follows that eigenvalue of a k-loop circuit (eigenvales of order n- k) and eigenvalues of the corresponding pure-loop circuit (eigenvalues of zeroth order) are related by inequalities (this case corresponds to imposing simultaneously or searially k constraints)

$$\lambda_j^{(n)} \le \lambda_j^{(n-k)} \le \lambda_{j=1}^{(n)} \qquad (8)$$

Inequalities (7) and (8) provide a simple technique of separating the roots of characteristic polynomials of operaors $Z^{(i)}(\lambda)$, which enables us to construct a simple effectivealgorithm of defining a full range of eigenvalues of an arbitrary LC-circuit with a great number of degrees of freedom (with a great numberof loops).

It should be noted that, as different from the traditional approach consisting in attempts to connect eigenfrequencies and partial ones, inequalities (7), (8) and the expression obtained for a Weinstein function (Proposition 3) make it possible to connect eigenfrequencies of individual impedances with eigenfrequencies of a k-loop circuit.

A question naturally arises what happens to eigenvalues in passing from the operator $Z^{(i)}(\lambda)$ to the operator $Z^{(i-1)}(\lambda)$. An answer is provided by the Aronszajn's lemma: when one constraint is imposed, the eigenvalue either may be preserved (and even its multiplicity may increase) or vanish (the latter corresponds to the case where the initial multiplicity equal to one decreases by one). Thisreasoning leads to

Definition 2. An eigenvalue of zeroth order is called i-conservative if it is preserved when i constraints are imposed and vanisheswhen i + 1 constraints are imposed. An eigenvalue of zeroth order $\lambda_j^{(n)}$ is called conservative if it is preserved when n-k constraints areimposed, i.e. it is an eigenvalue of both a pure loop circuit and a finitek-loop circuit.

A corollaryof the definition 2 is

Proposition 7. Eigenvalues of zero-th order of a pure loop circuit (of a base oscillatory system), the multiplicity m of which is greater than or equal to the number of node pairs in a k-loop circuit (to thenumber n-k of fictious loops), are conservative.

The proof of the proposition is almost obvious, and we omit it.

Proposition 7, seemingly so simple, proves to be rather effective, since when a primitive circuit has a sufficiently great number m of equal impedance (recall that their number is equal to the multiplicity of an eigenvalue of zeroth order), there is no need to calculate the corresponding eigenvalue of a k-loop circuit – it is enough only to verify the fulfilment of a simple inequality m > (n-k). The latter circumstance can be used in the synthesis of circuits with a given range of eigenfrequencies by simply choosing the required number of elements (impedances) of the same kind in a primitive circuit.

Conclusion

Weistain method of intermediate problems has been modified for a system with concentrated parameters – electrical circuits. Basis (initial) problem is defined as pure-loop (pure-node) circuit.

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A relationship between eigenvalues (proper frequencies) of impedances of separate branches of the circuit and loop impedances is established. A concrete forms for resolvents of circuit operators and corresponding Weinstein functions are obtained: the determinant of the conductance node matrix of an arbitrary k-loop LC-circuit is the Weinstein function for the loop impendance matrix of this circuit, and vice versa.

A simple technique of separating the roots of characteristic polynomials is elaborated. Finite steps recurrent process of intermediate problems of eigen values is determined. The latter leads to the important result conserning LC-circuit eigenvalues multiplicity, in the synthesis of circuits with a given range of eigenfrequencies by simply choosing the required number of elements (impedances) of the same kind in a primitive circuit.

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