Some Notes on the Majorants of Fourier Partial Sums in New Function Spaces

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Abstract

In this paper we present the boundedness criteria of majorants of partial sums of trigonometric and Walsh-Fourier series in weighted grand Lebesgue spaces.

Keywords: Grand Lebesgue space, Majorants, partial sums, norm convergence, trigonometric and Walsh-Fourier Series.

Introduction

Let $\mathrm{T}=(-\pi,\pi)$ and $1<p<\infty$, $\theta>0$. The weighted grand Lebesgue space of 2π-periodic functions is defined by the norm

$$\|f\|_{L_w^{p),\theta}} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^\theta}{2\pi}\int_T |f(x)|^{p-\varepsilon} w(x)dx\right)^{\frac{1}{p-\varepsilon}}$$

Here w is a weight function, i.e. an a.e. positive function which is integrable on T. When $w\equiv 1$, we set $L_w^{p),\theta}=L^{p),\theta}$

In the weighted case, grand Lebesgue spaces on the bounded sets of a Euclidean space were introduced by T. Iwaniec and C. Sbordone [1] for $\theta=1$, and by L. Greco, T. Iwaniec and C. Sbordone [2] for $\theta>1$. It is the well-established fact that these spaces are non-reflexive non-separable ones.

For the boundedness problems in $L_w^{p),\theta}$ for various integral operators we refer the readers to [3]-[6].

It is well known that the following continuous embeddings hold true:

$$L_w^p \to L_w^{p),\theta} \to L_w^{p-\varepsilon}, \quad 0<\varepsilon<p-1.$$

Let $f\in L^1(\mathrm{T})$ and

$$f(x)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)=\sum_{k=-\infty}^{\infty}c_k(f)e^{ikx} \quad \text{(eq.1)}$$

be its Fourier series. By $S_n(f,x)$ we denote the partial sums of (1).

A weight function is said to be of the Muckenhoupt class A_p $(1<p<\infty)$ if

$$\sup\left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w^{1-p'}(x)dx\right)^{p-1}<\infty \quad \text{(eq.2)}$$

where the supremum is taken over all intervals whose length is less than 2π.

Theorem1: Let $1<p<\infty$ and $\theta>0$. The following statements are equivalent:

i) There exists a positive constant $c_1>0$ such that

$$\left\|\sup_n|S_n(f,x)|\right\|_{L_w^{p),\theta}}\le c_4\|f\|_{L_w^{p),\theta}}$$

ii) $w\in A_p(\mathrm{T})$.

As has been mentioned above, the space $L^{p),\theta}$ is a non-separable one. The closure of L^p by the norm of $L^{p),\theta}$ does not coincide with the latter space.

We denote by $\dot{L}^{p),\theta}$ the closure of C^∞ with respect to the norm of $L^{p),\theta}$. As is known [7], $\dot{L}_w^{p),\theta}$ is a subspace of the space $L^{p),\theta}$ of functions satisfying

$$\lim_{\varepsilon\to 0}\left(\varepsilon^\theta\int_T |f(x)|^{p-\varepsilon} w(x)dx\right)^{\frac{1}{p-\varepsilon}}=0$$

.

Theorem 2: Let $1<p<\infty$ and $\theta>0$. Let $w\in A_p(\mathrm{T})$. Then for $f\in\dot{L}_w^{p),\theta}$ we have

$$\lim_{n\to\infty}\|S_n(f,\cdot)-f\|_{L_w^{p),\theta}}=0.$$

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Proofs of Theorem 1 and 2:

The sufficiency part of Theorem 1 it follows from the following known results:

Theorem A [8]. Let $1 < p < \infty$ and $w \in A_p(T)$.

Then the operator $f \mapsto S^*(f,x) = \sup_n |S_n(f,x)|$ is bounded in $L_w^p(T)$.

Here $$\|f\|_{L_w^p} = \left(\int_T |f(x)|^p w(x)dx \right)^{1/p}.$$

Theorem B [5, 6]: Let some operator T be bounded in $L_w^p(T)$ $(1 < p < \infty)$. Suppose also that T is bounded in $L_w^{p-\sigma}$ with the same weight w for some σ, $0 < \sigma < p-1$. Then T is bounded in $L^{p),\theta}$ for arbitrary θ, $0 < \theta < 1$.

Proposition C [9]:
Let $w \in A_p(T)$ $(1 < p < \infty)$
Then there exists some p_1, $1 < p_1 < p$, such that $w \in A_{p_1}(T)$.
Now we prove that in Theorem 1 i)⇒ii).
Let $I \subset (-\pi, \pi)$ such that $|I| \le \dfrac{\pi}{4}$. Let $f \ge 0$.
Let and suppose $f \subset I$.
Let a natural number is such that

$$\frac{\pi}{4(n+1)} \le |I| \le \frac{\pi}{4n}.$$

Estimating $S_n(f,x)$ for $x \in I$ we get

$$S_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt = \frac{1}{\pi}\int_I f(t)D_n(x-t)dt =$$

$$= \frac{1}{\pi}\int_I f(t)\frac{\sin\left(n+\frac{1}{2}\right)(x-t)}{\sin\frac{x-t}{2}}dt = cn\int_I f(t)dt \ge \ge \frac{c_1}{|I|}\int_I f(t)dt.$$

Then from i) we derive the estimate

$$\left\| \chi_I(x)\frac{1}{|I|}\int_I f(t)dt \right\|_{L_w^{p),\theta}(T)} \le c\|f\chi_I\|_{L_w^{p),\theta}(T)} \quad .(3)$$

Rewrite the latter inequality in the following form:

$$\|\chi_I\|_{L_w^{p),\theta}(T)} \cdot \frac{1}{|I|}\int_I f(t)dy \le c\|f\chi_I\|_{L^{p),\theta}(T)}.$$

Now we start to estimate of the right side.
We have

$$\|f\chi_I\|_{L_w^{p),\theta}(T)} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^\theta}{2\pi}\int_I |f(y)|^{p-\varepsilon} w(y)dy \right)^{\frac{1}{p-\varepsilon}}.$$

Applying the Hölder inequality we obtain

$$\|f\chi_I\|_{L_w^{p),\theta}(T)} \le \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^\theta}{2\pi}\right)^{\frac{1}{p-\varepsilon}}\left(\int_I f^p(y)w(y)dy\right)^{1/p}\left(\int_I w(y)dy\right)^{\frac{\varepsilon}{p(p-\varepsilon)}} =$$

$$= \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^\theta}{2\pi}\right)^{\frac{1}{p-\varepsilon}}\left(\int_I w(y)dy\right)^{\frac{1}{p-\varepsilon}}\left(\int_I w(y)dy\right)^{-\frac{1}{p}}\left(\int_I f^p(y)w(y)dy\right)^{1/p}$$

Thus from (3) follows

$$\frac{1}{|I|}\int_I f(y)dy\|\chi_I\|_{L_w^{p),\theta}(T)} \le c\|\chi_I\|_{L_w^{p),\theta}}\left(\int_I w(y)dy\right)^{-\frac{1}{p}}\left(\int_I f^p(y)w(y)dy\right)^{1/p}$$

Consequently,

$$\frac{1}{|I|}\int_I f(y)dy \le c\left(\int_I w(y)dy\right)^{-\frac{1}{p}}\left(\int_I f^p(y)w(y)dy\right)^{1/p}$$

In the latter inequality substitute the test function $f(y) = w^{-\frac{1}{p-1}}(y)\chi_I(y)$. We get. We get

$$\left(\int_I w(y)dy\right)^{1/p}\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}(y)dy \le c_1\left(\int_I w^{-\frac{p}{p-1}}(y)w(y)dy\right)^{1/p}.$$

$$\frac{1}{|I|}\left(\int_I w(y)dy\right)^{1/p}\left(\int_I w^{-\frac{1}{p-1}}(y)dy\right) \le c_1\left(\int_I w^{-\frac{1}{p-1}}(y)dy\right)^{1/p}$$

with a constant c_1 independent of I.

We conclude that $w \in A_p(T)$.

The proof of Theorem 2 is standard (see [10], Chapter 7, Section 6) applying the following

Proposition 1: Let $1 < p < \infty$ and $\theta > 0$.. Then the conjugate function

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$$\widetilde{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \frac{f(x+t)-f(x-t)}{2tg\frac{t}{2}}dt$$

is bounded in $L_w^{p),\theta}$ if and only if $w \in A_p(T)$.

The sufficiency part of Proposition 3 follows from Theorem B and the well-known result of R. A. Hunt, B. Muckenhoupt and R. L. Wheeden [11] that operator $\widetilde{f}$ is bounded in L_w^p if and only if $w \in A_p$.

The necessity part of proposition 3 we can prove analogously as for the Hilbert transform [4].Note that Theorems similar to the Theorems 1 and 2 are valid for Walsh-Fourier series as well. The Walsh-Fourier series $\sum_{k\geq 0} a_k W_k(x)$ is a dyadic analogue for the Fourier series.

The Walsh functions $W_0(x), W_1(x), \ldots$ are supported in $[0,1)$ and can be defined:

$W_0(x) = \chi_{(0,1)}$, and for even and odd integers,

$$W_{2n}(x) = W_n(2x)\chi_{\left(0,\frac{1}{2}\right)}(x) + W_n(2x-1)\chi_{\left(\frac{1}{2},1\right)}(x), \quad n \geq 1$$

$$W_{2n+1}(x) = W_n(2x)\chi_{\left[0,\frac{1}{2}\right)}(x) - W_n(2x-1)\chi_{\left[\frac{1}{2},1\right)}(x), \quad n \geq 0.$$

By A_p we define the class of weights by the condition (2) holds over all dyadic intervals.

For the functions from $L_w^p[0,1)$ it is known the following result

Theorem D [12]: For any $1 < p < \infty$ and $w \in A_p^d$, it holds that

$$\left\| \sup_n \left| \sum_{0\leq k\leq n} a_k W_k(\cdot) \right| \right\|_{L_w^p} \leq c\|f\|_{L_w^p}$$

with a constant independent of $f \in L_w^p$.

We have the following statements:

Theorem 5: For any $1 < p < \infty$ and $\theta > 0$, and $w \in A_p^d$ then the following two conditions are equivalent:

i) $$\left\| \sup_n \left| \sum_{0\leq k\leq n} a_k W_k(\cdot) \right| \right\|_{L_w^{p),\theta}} \leq c\|f\|_{L_w^{p),\theta}}$$

and

ii) $w \in A_p^d$.

Here by a_k are denoted the Walsh-Fourier coefficients of f.

Theorem 6: For any $1 < p < \infty$ and $\theta > 0$, and $w \in A_p^d$ and for arbitrary $f \in L_w^{p),\theta}$ we have

$$\lim_{m\to\infty} \left\| f - \sum_{0\leq k\leq n} a_k W_k \right\|_{L_w^{p),\theta}} = 0$$

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