Spinor Representation of Spatial Rotation Group for Rigid Bodies

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Abstract

The method of spinor representation of three-dimensional generalized rotations have been introduced. Relations between the parameters of the spinor representation of three-dimensional generalized rotations group and the coordinates of the initial and terminal points of rotation have been obtained. The simple relations between the elements of a three-dimensional orthogonal matrix of the basic representation and the Euler angles, on the one hand, and the coordinates of the initial and terminal points of rotation, on the other hand were derived.

Keywords: rigid body, rotation, Euler angle, Spinor, matrix transformation.

Introduction

It is well known from Euler's theorem that the general finite displacement of a rigid body, that has one point fixed, can be represented as a rotation through a certain angle about some straight line passing through the point [1]. The body requires three parameters for the description of its pose (orientation) relative to some datum. Thus a three component vector can represent a single finite angular displacement about a fixed point in 3D. This means that any two rotations of arbitrary magnitude about different axes can always be combined into a single rotation about some axis.

At first sight, it seems that we should be able to express a rotation as a vector which has a direction along the axis of rotation and a magnitude that is equal to the angle of rotation. Unfortunately, if we consider two such rotation vectors, θ1 and θ2, not only would the combined rotation θ be different from θ1 + θ2, but in general θ1 + θ2 ≠θ2 + θ1.It is clear that the result of applying the rotation in x first and then in y is different from the result obtained by rotating first in y and then in x. Therefore, it is clear that finite rotations cannot be treated as vectors, since they do not satisfy simple vector operations such as the parallelogram vector addition law.

Many different approaches to represent a finite rotation about an axis have been explored.

These include real orthogonal 3x3 matrices, Euler angles, special unitary 2x2 matrices, unit quaternions. etc.[2]. The commonest representation of 3D rotations in the engineering field is the real orthogonal 3x3matrix [2]. This has nine elements, of which only three are independent, and one relatively minor difficulty with this approach arises in trying to relate the nine matrix elements to the axis of rotation and the angle turned through. The following real orthogonal 3x3 matrix represents a rotation of a rigid body through an angle θ about a general line through the origin with direction cosines (l, m, n):

$$\mathbf{R}_{(l,m,n)}(\theta)=\begin{bmatrix} l^2(1-\cos\theta)+\cos\theta & lm(1-\cos\theta)-n\sin\theta & ln(1-\cos\theta)+m\sin\theta \\ ml(1-\cos\theta)+n\sin\theta & m^2(1-\cos\theta)+\cos\theta & mn(1-\cos\theta)-l\sin\theta \\ nl(1-\cos\theta)-m\sin\theta & nm(1-\cos\theta)+l\sin\theta & n^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

A rotation of a rigid body can be also uniquely defined by rotating the rigid body 3 times in succession about any 3 non-planar directions. These rotation angles φ, θ and ψ are called the Eulerian angles. The matrix of the rotation is:

$$A=\begin{Vmatrix} \cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi & -\cos\varphi\sin\psi-\cos\theta\sin\varphi\sin\psi & \sin\varphi\sin\theta \\ \sin\varphi\cos\psi+\cos\theta\cos\varphi\sin\psi & -\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi & -\cos\varphi\sin\theta \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{Vmatrix}$$

where $-\pi<\varphi\leq\pi$, $0\leq\theta\leq\pi$ and $-\pi<\psi\leq\pi$ are Euler angles.

Given the matrices with its nine elements, a relatively lengthy algebraic process is required to

Determine the direction cosines of the axis, the angle of rotation or Euler angles. Also, particularly in the field of kinematics, the result of compounding several (often many) matrices in succession involves large amount of computations.

The basic problem arising in this context can be formulated as follows: given two three-dimensional points $x(x^1, x2, x3)$ and $y(y^1,y^2,y^3)$, it is required to define the set of all possible transformations and centers of rotations which bring about the transformation of the point x to the point

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y. It is obvious that this problem can be easily extended to the case where instead of two points we consider two finite sets of points $\{x_i\ (x_i^1, x_i^2, x_i^3\}$ and $\{y_i\ (y_i^1, y_i^2, y_i^3\}$ i=1, 2,…m, which corresponds to the case of rotations of a rigid body.

Statement of the Problem

Let L^3 be a linear Euclidean space with orthonormalized basis vectors e_1, e_2, e_3. To each vector $x=x^1e_1+x^2e_2+x^3e_3$ of the space L^3 we assign a traceless Hermitian matrix

$$X = \begin{vmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{vmatrix}, \tag{1}$$

whose elements are the so-called spinor components of the vector x. When we pass from the usual Euclidean components of the vector x to the spinor ones, we thereby identify the vector x with Hermitian functionals on the two-dimensional linear space C^2 over the field of complex numbers C. Denote by L (C^2) the set of all Hermitian functionals on C^2 which is a linear three-dimensional space over the field of real numbers provided that Pauli matrices are taken as basic elements. Then for each matrix of form (1) we have the decomposition

$$X= x^1\sigma_1+ x^2\sigma_2+x^3\sigma_3, \tag{2}$$

where

$$\sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

are Pauli matrices.

From decomposition (2) it follows that the set L(C^2) is a linear three-dimensional space over the field of real numbers and thus it can be identified with L^3. Note that to each basis vector of the two-dimensional space C^2 we can assign the basis vectors $\sigma_1,\sigma_2,\sigma_3$ of the space L(C^2) (and also the orthonormalized basis vectors e_1,e_2,e_3 due to the identification of L^3 and L(C^2)): each of the matrices σ_i is represented as some linear combination of tensor products of basis vectors of the space C^2 [3]. The foregoing reasoning implies that for any matrix $C \in C^2$,, which is a matrix of transformation between two basis vectors of the space C^2, there also exists a transformation matrix of the corresponding orthonormalized basis vectors in the space L^3.

Proposition 1. The matrix of transformation of the basis vectors in C^2 is unitary (3)

The problem can be now reformulated in terms of the spinor space C^2: Given two traceless matrices of Hermitian functionals

$$X = \begin{vmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{vmatrix} \text{ and } Y = \begin{vmatrix} y^3 & y^1 - iy^2 \\ y^1 + iy^2 & -y^3 \end{vmatrix},$$

it is required to define:

1) A set of unitary matrices $C = \begin{vmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{vmatrix}$ which satisfy the equality

$$Y = \overline{C}^T XC; \tag{4}$$

2) One-dimensional subspaces which are invariant with respect to transformations represented by matrices C (i.e. a set of respective rotation centers).

Note that since the transformation C is unitary, the vector norms defined by the determinants of matrices of the Hermitian functionals X and Y coincide and therefore (4) defines rotation.

From equality (4) we can obtain the following system of linear homogeneous equations with respect to the unknown variables α and β:

$$x_3\alpha + \gamma\beta = y_3\alpha - \overline{\delta\beta}$$

$$\overline{\gamma}\alpha - x_3\beta = y_3\beta + \overline{\delta}\overline{\alpha}, \tag{5}$$

where $\gamma = x_1 + ix_2$ and $\delta = y_1 + iy_2$.

For arbitrary α, a solution of (5) is given by

$$\beta = \frac{\overline{\gamma}\alpha - \overline{\delta}\overline{\alpha}}{x_3 + y_3}. \tag{6}$$

From (6) we have

$$\operatorname{Re}\beta = \beta_1 = \frac{\alpha_1(x_1 - y_1) + \alpha_2(x_2 + y_2)}{x_3 + y_3} \text{ and}$$

$$\operatorname{Im}\beta = \beta_2 = \frac{\alpha_2(x_1 + y_1) - \alpha_1(x_2 - y_2)}{x_3 + y_3}. \tag{7}$$

Using the unitarity of the matrix

C ($\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$), we can define either $\alpha_1 = \operatorname{Re}\alpha$ or $\alpha_2 = \operatorname{Im}\alpha$ Note that one of these parameters remains arbitrary. Thus (6) defines rotation for $\alpha \neq 0$ and $x_3 + y_3 \neq 0$.

The invariance of the rotation center Z (z_1, z_2, z_3) with respect to the transformation C is written as a condition

$\overline{C}^T ZC = Z$ whence we obtain

$$z_3\alpha + \mu\beta = z_3\alpha - \overline{\mu}\beta$$

$$\overline{\mu}\alpha - z_3\beta = z_3\beta + \overline{\mu}\overline{\alpha},$$

where $\mu = z_1 + iz_2$.

The latter formula leads to the system

$$\beta_1 z_1 - \beta_2 z_2 = 0;$$

$$\alpha_2 z_2 - \beta_1 z_3 = 0;$$

$$\alpha_1 z_1 - \beta_2 z_3 = 0.$$

It is not difficult to verify that the determinant of this system considered for the unknown values z_1, z_2 and z_3 is identically zero and therefore for given α_1, α_2, β_1 and β_2 ($\alpha \neq 0$ and $x_3 + y_3 \neq 0$) there always exist nontrivial solutions written in the form

$$z_1 = \frac{\beta_2}{\alpha_2} z_3;\ z_2 = \frac{\beta_1}{\alpha_2} z_3, \tag{8}$$

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where z_3 is arbitrary. Equations (7) define the one-parametric set of transformations Ct due to which (x^1, x^2, x^3) changes to (y^1, y^2, y^3) by means of rotation. If we choose α_1 as a parameter, then to its each fixed value defining the unique transformation $C_{t=\alpha_1}$ we can assign the set of rotation centers lying in the plane

$$\alpha_2 z_1 + \alpha_2 z_2 - (\beta_1 + \beta_2) z_3 = 0 \tag{9}$$

whose equation is readily obtained from (8).

Thus, (7) together with the normalization condition define a generalized rotation transforming (x^1,x^2,x^3) to (y^1,y^2,y^3) with respect to the set of centers which is defined by (9).

Relations between Transformations in C^2 and L^3

We can establish the correspondence between the elements of the transformation matrix $C = \begin{vmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{vmatrix}$ in C^2 and the elements of the orthogonal real matrix of rotation A in L^3.

The matrix A is, by definition, the matrix of transformation between two orthonormalized basis vectors of the space L^3 and its rows are decompositions of the new basis vectors in terms of the initial basis vectors. Hence due to the identification of the spaces L (C^2) and L^3 we have

$$\overline{C}^T \sigma_i C = a_i^{i'} \sigma_{i'} \quad (i, i' = 1,2,3), \tag{10}$$

Where σ_i are the Pauli matrices corresponding to the initial basis, $\sigma_{i'}$ are the Pauli matrices of the new basis, and $\alpha_i^{i'}$ are the elements of the matrix A^{-1}.

Formula (10) can be written explicitly in the form of three matrix equalities

$$a_1^1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + a_2^1 \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} + a_3^1 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{vmatrix} * \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} * \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix},$$

$$a_1^2 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + a_2^2 \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} + a_3^2 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{vmatrix} * \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} * \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix}$$

$$a_1^3 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + a_2^3 \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} + a_3^3 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = \begin{vmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{vmatrix} * \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} * \begin{vmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{vmatrix}$$

which readily yield the following expressions for calculating the elements of the matrix A by the elements of the matrix C:

$$a_1^1 = (\alpha_1^2 - \alpha_2^2) - (\beta_1^2 - \beta_2^2); \quad a_2^1 = 2(\alpha_1 \alpha_2 + \beta_1 \beta_2) \quad a_3^1 = 2(\alpha_2 \beta_2 - \alpha_1 \beta_1);$$
$$a_1^2 = 2(\beta_1 \beta_2 - \alpha_1 \alpha_2); \quad a_2^2 = (\alpha_1^2 - \alpha_2^2) + (\beta_1^2 - \beta_2^2); \quad a_3^2 = 2(\alpha_1 \beta_2 + \alpha_2 \beta_1);$$
$$a_1^3 = 2(\alpha_1 \beta_1 + \alpha_2 \beta_2); \quad a_3^2 = 2(\alpha_2 \beta_1 - \alpha_1 \beta_2); \quad a_3^3 = (\alpha_1^2 + \alpha_2^2) - (\beta_1^2 + \beta_2^2).$$

(11)

Expressions (11) enable us to calculate the element of the matrix A through the given coordinates of three points (initial, terminal and the center and which define rotation):

On the other hand, taking into account that the matrix A can be written in the form [1]

$$A = \begin{Vmatrix} \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi & -\cos\varphi\sin\psi - \cos\theta\sin\varphi\sin\psi & \sin\varphi\sin\theta \\ \sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi & -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi & -\cos\varphi\sin\theta \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{Vmatrix}$$

(12)

where $-\pi<\varphi\leq\pi$, $0\leq\theta\leq\pi$ and $-\pi<\psi\leq\pi$ are Euler angles, it easily follows that expressions (11) enable us to define Euler angles as well

$$\cos\theta = a_{33}; \ \sin\varphi\sin\theta = a_{31} \text{ and } \sin\psi\sin\theta = a_{13} \tag{13}$$

Thus, the spinor approach to the representation of rotations of a three-dimensional space makes it possible not only to describe sets of possible representations of rotations (which we call generalized rotations), but also to consider matrices of real orthogonal rotations in L^3 as particular cases of transformations in C, which leads to obtaining simple formulas for calculating elements of real orthogonal matrices and such important technical parameters as Euler angles. It is significant that all the literature explains how to rotate after the Euler angles are given. There is no clear explanation for calculating the Euler angles. The latter is very important problem in engineering, computer graphics and simulation. The method considered allows us to calculate Euler angles in case when a general rotation angle between initial and final positions of rigid body is given

The results obtained above it give possibility to calculate easily Euler angles ensuring the turn of a point $x(x^1, x^2, x^3)$ into a point $y(y^1, y^2, y^3)$. If we assume that zero Euler angles $x(x^1, x^2, x^3)$ correspond to the initial point, then the control of rotation consists in changes in time of Euler angles $\theta_0 = \phi_0 = \psi_0 = 0$ from initial values $\theta_0; \phi_0; \psi_0$ to final ones $\theta_f; \phi_f; \psi_f$ computed by the formulas (13). In a general form the control process can be presented as functions of change of Euler angles $\theta(t); \phi(t); \psi(t)$ that should satisfy the following conditions:

$$\theta(t_0) = 0; \ \phi(t_0) = 0; \ \psi(t_0) = 0,$$

$$\theta(t_f) = \theta_f; \ \phi(t_f) = \phi_f; \ \psi(t_f) = \psi_f, \tag{14}$$

where t_0 and t_f - initial and final moments of control process.

Based on the above-said the problem of determination of control functions $\theta(t)$; $\phi(t)$; $\psi(t)$ naturally follows, to which the given work is devoted.

It is necessary to point out that the following: dependences $\theta(t)$; $\phi(t)$; $\psi(t)$ have kinematical nature character, as they do not allow for neither moments nor elasticity nor any other dynamic characteristics of process, therefore after their definition the task of synthesis of the dynamic adaptive control on the basis of these functions arises [3]. This problem will be considered in the subsequent works.

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Conclusion

The obtained results permit to reduce actually three-dimensional problem of spatial motion control to the one-dimensional problem. The simple method to calculate such important technical parameters as Euler angles is proposed.

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