Journal of Technical Science and Technologies, 1(2):5-10,2012 ISSN:2298-0032

Symbolic Computation Techniques for Two-Dimensional Electrostatics with Maxima

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Abstract

Computer algebra systems (CAS) are in the agenda of today's educational systems and engineering world. Open source CAS, like Maxima are evaluated as an alternative to the commercial ones that are being improved according to changing needs of engineers and educators. The main problem with Maxima is "how to use it in the correct way?" We witness a shortage of sample studies showing the effective use of Maxima and insufficiencies in its user's manual. In this paper, the abilities of Maxima are presented with four worksheets including four commonly used examples in two-dimensional electrostatic applications.

Keywords: Open Source CAS, Symbolic computation, Maxima worksheet, two dimensional electrostatic.

Computer Algebra Systems

A Computer Algebra System (CAS) is a type of software package that is used in the manipulation of mathematical formulae. The main purpose of a CAS is to automate long, boring, and sometimes difficult algebraic manipulation tasks. The principal difference between a CAS and a calculator is the ability to deal with equations symbolically rather than numerically. The specific uses and capabilities of these systems changes from one system to another .But, the main goal remains the same: manipulation of symbolic equations. CAS often includes facilities for graphing equations and provides a programming language for the user to define his/her own procedures. (Pierce & Stacey 2004).

CAS were discovered and used by the scientists in their own researches. Since 1960, many of them were developed for the specific purposes in research laboratories. For example MACSYMA was developed and used in MIT. CASs can be classified in two groups, commercial ones such as Matematica, Matlab and Maple and the other group is open source programs such as Maxima, Scilab, Reduce, and SAGE.

The integration of CAS in to the educational systems is a long process. For example, MAPLE which is a trademark and developed by Waterloo Maple inc. has published the 15th version of the program in 2012. As the number of versions increases, the details needed by users, especially students and teachers, with the exception of engineers increases. The price of only Student Package is 124 USD. Matlab is another CAS with a very large library for any type of users. It appeared in the 1980's. Its latest version is Matlab 7.7 R2008b.

Besides the commercial ones, there are many open source CAS developed by volunteers from many countries. These programs are available at http://www.opensource-math.org/opensource_math.html. They are found "working and good" for their efforts. For example in the SAGE's website they explain their mission as "Mission: Creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab."

A maximum is one of the famous CAS Packages. It is practical to download and use. It has a fast and good interface for users. Its free version is MACSYMA. It is being developed by volunteers. It has approximately 50.000 users all over the world.

Open source programs are an opportunity for developing countries and universities or any educational institutions without paying for software.

1. CAS and Education

As the CAS technology got cheaper and available, these systems are included in the education system. CAS has added new components to mathematics education. Engineering and mathematic based sciences are affected from such computer systems at an increasing rate as the days pass. Many universities have added computer algebra lessons in their curricula.

It is a well-known reality that computer aided instruction is a good "style" to teach. Computerized learning environment is accepted as "attractive and assists to learning process" (Garcia, Galan. et all, 2009) by the students and educators. Beyond the advantages CAS, have some question marks on them.

Although they are "highly valuable" (Galbraith and Pemberton, 2002), the main problem with them is "how

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to use them effectively?" (Pierce & Stacey 2004). They are seen as a "black box" (Picard, 2006), for every student and lecturers who do not know about the software. The problems with CASs are detailed by the searches. The first one is the unfamiliarity of the users with hardware and software. (Wood and D'Souza, 2007) The second one is the cost of the availability of the technology.

A CAS can be used in two different ways. One of them is a "tutorial" to teach the "subject". The other one is to make the users ready for new searches. The main role at this stage is the lecturers. The lecturer must learn the CAS, and then prepare the material that he/she teaches and then presents to his/her students. CAS need the help of the lecturers for becoming a "gray or a white box" to join the educational processes in the classrooms. (Allen et all, 1999, Picardi, 2007).

In this study, we present a simple "tutorial" for the lecturers and the students with 'Maxima' on the subject of "Two dimensional electrostatics".

2. Two Dimensional Electrostatics

The Laplace's equation is, perhaps, the most important partial differential equation in applied mathematics. It can be solved by separation of variables, numeric methods and by complex function theory.

This method is very useful in calculating the distribution of vector fields such as;

1. An electrostatic field between parallel conductors of having various shapes,
2. An electric current in a uniform conducting sheet,
3. The magnetic field around a straight conductor carrying current in the neighborhood of parallel ferromagnetic masses. Two dimensional electrostatic fields is produced by a system including two parallel charged wires, plates and cylindrical conductors which are parallel to the z-plane in a (x,y,z) dimension system.

"Potentials in physics" are described by the solutions to Laplace equation

$$\nabla^2 \Phi = 0 \tag{1}$$

The solutions are called as harmonic functions if they have continuous 2nd partial derivatives.

If any function given as a complex function of $f(z)=w=u+iv$ is an analytical function of the complex variable $z=x+iy$ and

$$f(z) = w(x, y) + iv(x, y) \tag{2}$$

The real part u and the imaginary part of "v" of w(z) satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3}$$

Then the Laplace equations in two dimensions are written as;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ and \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{4}$$

It's easily seen that the real and imaginary parts of w(z) satisfy the Laplace's differential equations in two dimensions. The functions **u** and **v** are called conjugate functions.

The real and imaginary parts of a complex analytical function

$$F(z = x + iy) = \Phi(x, y) + i\psi(x, y) \tag{5}$$

are satisfied with 2-D Laplace's equation as we explained above

$$\nabla^2 \Phi = \nabla^2 \Psi = 0$$

If Φ(x,y) represents the potential function, by working with f(z) we can handle both equi-potential lines (Φ=constant) and the lines of flux (Ψ=constant).

As we know that electric field intensity is defined as the coulomb's force acting on a unit charge replace at (x,y) point and it's derived from (Φ) , the potential (function), a scalar field as

$$\vec{\boldsymbol{E}}(x, y) = -grad(\boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y})) = -\frac{\partial^2}{\partial x^2}\Phi(x, y) - \frac{\partial^2}{\partial y^2}\Phi(x, y) \tag{6}$$

Φ(x,y) is written is complex form

$$\Phi(x, y) = \Phi_x(x, y) + i\Phi_y(x, y) \tag{7}$$

The complex potential function of f(z) on z is written as

$$f(z) = \Phi(x, y) + i\Psi(x, y) \tag{8}$$

The curves $\Phi(x, y) = K_1$ are called the equipotential curves and the curves $\Psi(x, y) = K_2$ are called as the lines of flux. When a small test charged is set free in the field E then it will move along a line of flux. Boundary value problems for the potential function ***φ(x, y)*** are solved by **Maxima** as below.

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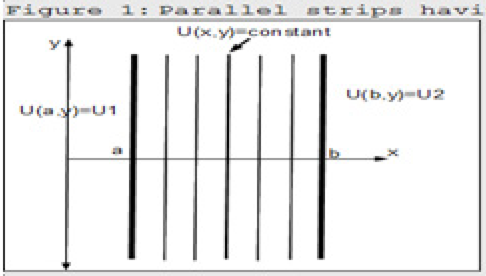
Example1:
Find the function u(x,y) that is harmonic in the vertical strip and takes on the boundary values
U(a,y1)=U1 for all y, and
U(b,y2)=U2 for all y,
along the vertical lines x=a and x=b, respectively.

Figure 1: Parallel strips having the potentials U1 and U2 respectively



Solution with Laplace's Equation:
We need a solution that has on constant values along the vertical lines from x=x0 and that U(x,y) be a function of x alone. It is U(x,y)=P(x) for a<= x =<b and for all y.
Laplace's equation Uxx(x,y)+Uyy(x,y)=0 implies that P''(x)=0, which implies P(x)=mx+c, where m and c are constants. The stated boundary conditions U(a,y)=P(a)=U1 and U(b,y)=P(b)=U2 lead to the solution: (as done below)

(%i1) `q:'diff(P(x,y),x,2)=0;`

(%o1) $\frac{d^2}{d x^2} P(x, y)=0$

(%i2) `ode2(q,P(x,y),x);`

(%o2) $P(x, y)= \%k2\, x + \%k1$

(%i3) `bc2(%,x=(a),P(a,y)=U1,x=(b),P(b,y)=U2);`

(%o3) $P(x, y)=\frac{x\,(U1-U2)}{a-b}-\frac{b\,U1-a\,U2}{a-b}$

(%i4) `P(x,y):=(x*(U1-U2))/(a-b)-(b*U1-a*U2)/(a-b);`

(%o4) $P(x, y):=\frac{x\,(U1-U2)}{a-b}-\frac{b\,U1-a\,U2}{a-b}$

(%i5) `load(vect)$`

(%i6) `express(grad(P(x,y)));`

(%o6) $[\frac{d}{d x}\left(\frac{x\,(U1-U2)}{a-b}-\frac{b\,U1-a\,U2}{a-b}\right), \frac{d}{d y}\left(\frac{x\,(U1-U2)}{a-b}-\frac{b\,U1-a\,U2}{a-b}\right), \frac{d}{d z}\left(\frac{x\,(U1-U2)}{a-b}-\frac{b\,U1-a\,U2}{a-b}\right)]$

(%i7) `El_field:ev(%,diff);`

(%o7) $[\frac{U1-U2}{a-b}, 0, 0]$

(%i8) `U1:-10;U2:10;a:-2;b:2;`

(%o8) -10

(%o9) 10

(%o10) -2

(%o11) 2

(%i12) `potentials:ev(P(x,y));`

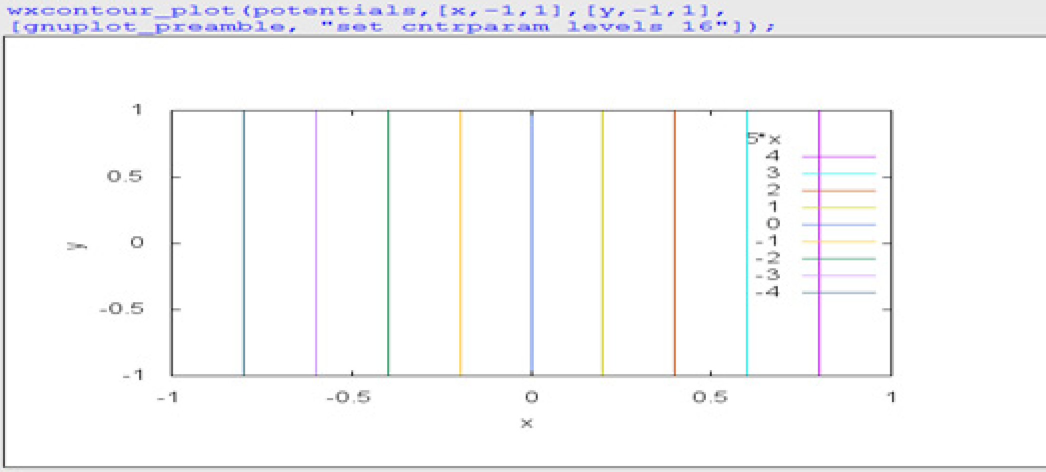
(%o12) $5\,x$

(%i13) `elfield:ev(El_field);`

(%o13) [5, 0, 0]

(%i14) `wxcontour_plot(potentials,[x,-1,1],[y,-1,1], [gnuplot_preamble, "set cntrparam levels 16"]);`

(%t14)



(%o14)

Frame1: Parallel strips having the potentials u1 and u2 respectively.

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Example2: Find the electrical potential phi(x,y) in the region between two infinite coaxial cylinders r1=a and r2=b, which are kept at potentials U1 and U2, respectively and illustrate the curves.

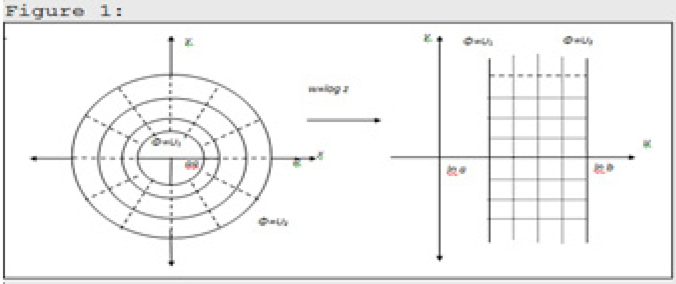
Solution:The transformation w=u+iv=log(z)=log(x+iy) maps the annular region between the circles r1=a, and r2=b onto the infinite strip ln a<u<ln b in the w plane, as shown in the figure 2.
The potential phi(u,v) in the infinite strip has the boundary values

phi(log a,v)=U1 and phi(log b,v)=U2 for all v

The electrical potential can be written as

Figure 2:Potential lines in a coaxial cable

Figure 1:



```
(%i1) phi(u,v):=U1+((U2-U1))/(log(a)-log(b))*(log(z)-log(a));
```

(%o1) $\phi(u,v):=U1+\frac{U2-U1}{\log(a)-\log(b)}(\log(z)-\log(a))$

The equipotential curves U(x,y)=constant are concentric circles centered on the origin, and the lines of flux are portions of rays emanating from the origin.
If U2<U1,then the situation is as illustrated in Figure 3.
The equipotential curves can be plotted by Maxima as below:
Let us assume that U1=0 volts, r1=0.01 m ,U2=40 volts and r2=0.02 m

```
(%i2) fpprintprec:4$
(%i3) phi(u,v):=U1+((U2-U1))/(log(r2)-log(r1))*(log(u^2+v^2)-log(r1));
```

(%o3) $\phi(u,v):=U1+\frac{U2-U1}{\log(r2)-\log(r1)}\left(\log(u^2+v^2)-\log(r1)\right)$

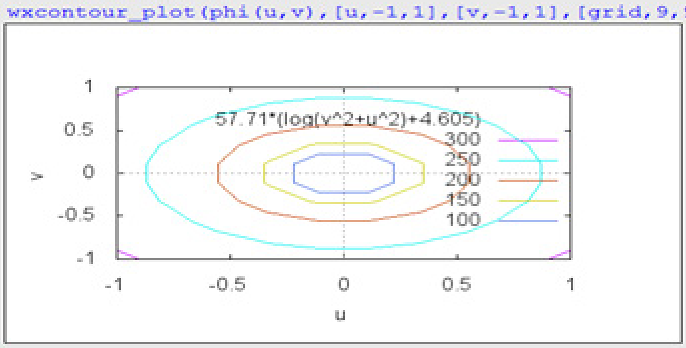
```
(%i4) U1:0;r1:0.01;U2:40;r2:0.02;
(%o4) 0
 (%o5) 0.01
 (%o6) 40
 (%o7) 0.02
(%i8) phi(u,v);
```

(%o8) $57.71\left(\log(v^2+u^2)+4.605\right)$

Figure 3: Equipotential lines:

```
(%i9) wxcontour_plot(phi(u,v),[u,-1,1],[v,-1,1],[grid,9,9]);
```

(%t9)



(%o9)

Frame 2: second example about finding electrical potentials.

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```
Example 3: Find the electrical potential phi (x,y)produced by two charged half-planes
that are perpendicular to the z-plane and pass through the rays x<-1, y=0 and x>1, y=0
where the planes are kept at the fixed potentials and illustrate the potential lines.
phi(x,0)=-10 for x<-1
phi(x,0)= 10 fpr x>1
Solution:
w=u+iv=>arcsin(z)=arcsin(x+iy)
is a conformal mapping of the z-plane slit along the two rays x<-1, y=0 and x>1, y=0
on to the vertical strip -pi/2 < u < pi/2. The new problem is to finr the potential
phi (u,v) that satisfies the boundary values of

phi(-pi/2,v)=-10 for all v and phi (pi/2,v)=10 for all v; by using
(%i1) phi(x,y)=U1+((U2-U1)/(b-a))*(x-a);
```

(%o1) $\phi(x,y)=\frac{(x-a)(U2-U1)}{b-a}+U1$

```
we get
(%i2) phi(u,v)=(600)*u/(%pi);
```

(%o2) $\phi(u,v)=\frac{600\,u}{\pi}$

```
The solution in the z-plane is
```

$$\phi(x,y)=\frac{u}{\pi}\,Re\,arcsin(z)=\frac{u}{\pi}.arcsin\left(\frac{\sqrt{(x+1)^2+y^2}-\sqrt{(x-1)^2+y^2}}{2}\right)$$

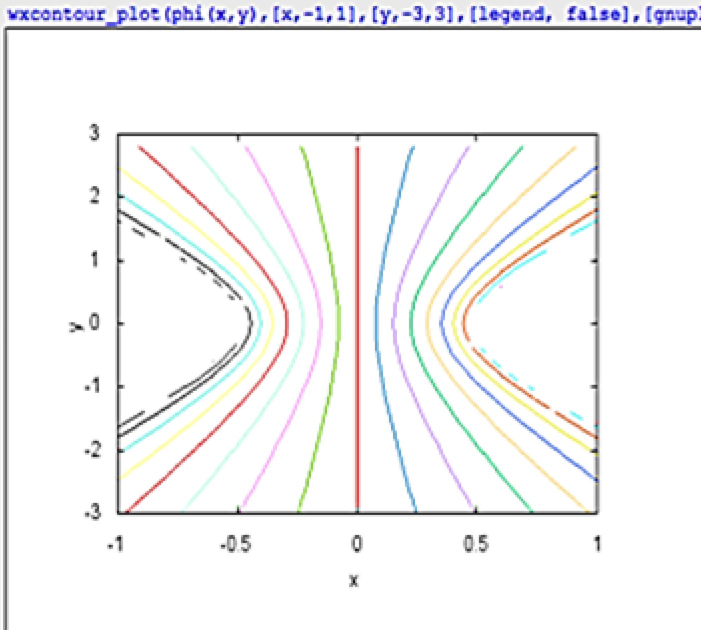
```
Equipotential curves are shown by Maxima as below:

(%i16) phi(x,y):=((-10)*asin(-sqrt((x+1)^2+y^2)+sqrt((x-1)^2+y^2)))/%pi;
```

(%o16) $\phi(x,y):=\frac{(-10)\,asin\left(-\sqrt{(x+1)^2+y^2}+\sqrt{(x-1)^2+y^2}\right)}{\pi}$

```
(%i40) wxcontour_plot(phi(x,y),[x,-1,1],[y,-3,3],[legend, false],[gnuplot_preamble, "set cntrparam levels 20"]);
```

(%t40)



(%o40)

Frame 3: third example about finding electrical potentials.

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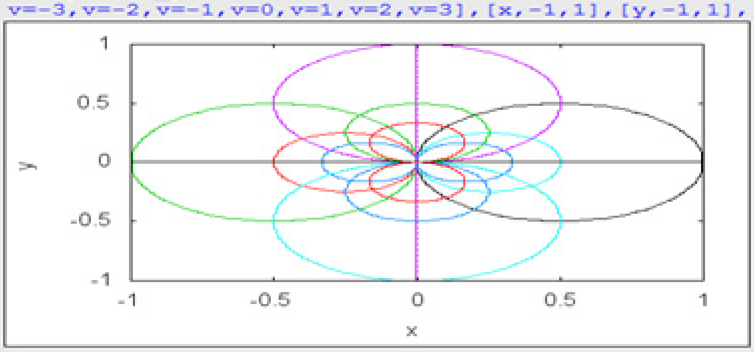
```
Example 4: Two charges are replaced on points p1(1,1)and p2(-1,1).
Illustrate the field lines in x-y plane.
Solution: The solution with complex analysis is done as below.
(%i1) z:x+%i*y;
(%o1) %i y+x
(%i2) f:1/z;
(%o2) 1/(%i y+x)
(%i3) u:realpart(f);v:imagpart(f);
(%o3) x/(y^2+x^2)
(%o4) -y/(y^2+x^2)
(%i5) load(implicit_plot)$
(%i6) wximplicit_plot([u=-3,u=-2,u=-1,u=0,u=1,u=2,u=3,
v=-3,v=-2,v=-1,v=0,v=1,v=2,v=3],[x,-1,1],[y,-1,1],[legend,false]);
(%t6)
```




```
(%o6)
```

Frame 4: fourth example illustrating the field lines in x-y plane

Discussion:

Maxima's mathematical performance and graphical illustrations are sufficient for the users to prepare computer worksheets; inserting texts and other images. These abilities make Maxima ***'applicable'*** in computer lab sessions of engineering courses. But the main deficiency with Maxima and other open source CAS still exists; they don't have sufficient and encouraging 'user's manuals'. Commercial ones, with their supportive tutorials are better than a mathematical tool; they can be accepted as an instructional tool. On the other hand, open source ones are still as 'black boxes'. These systems must be 'improved with sample studies showing the effective and correct way

of using them by the volunteers that develop these systems.

Conclusion:

In this paper, we presented a sample use of Maxima in four worksheets including common examples for undergraduate students of electrostatic. We saw that Maxima is a 'sufficient tool' for solving 'two dimensional electrostatic problem

Computer Algebra Systems are valuable tools for in-class instructional processes. It can enable lecturers make their lectures more attractive. With these tools, many examples can be done with a few clicks in the short period of the lessons. However, the first and main mission is the lecturers' mission; preparation of the worksheets before the lesson.

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