Spherical Spline Solution of the Heat Equation

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Abstract

We consider a spline approximation to the solution of the heat equation on the unit sphere. The time derivative is approximated by a backward divided difference resulting in an implicit method of order two. The differential equation on the sphere is solved by means of spherical splines. Numerical experiments exhibit quadratic convergence for the time variable and at least quadratic convergence with respect to the spacial variables. Spherical harmonics approximation is considered for the purpose of comparison with splines.

Keywords: Spherical Splines, Spherical Harmonics, Heat Equation

1. Introduction

Polynomial splines are becoming an increasingly popular tool in approximating solutions of partial differential equations (PDE) (Awanou et al. 2006; Awanou and Lai 2005; Aziz et al. 2005; Dehghan and Lakestani 2007; Gardner et al. 2007; Hu et al. 2007; Lai et al. 2002; Lai et al. 2003; Lai et al. 2004; Lai and Wenston 2004, Speleers et al 2006). Spherical homogeneous splines have demonstrated properties analogous to bivariate splines (Alfeld et al. 1996; Baramidze and Lai 2005; Baramidze et al. 2006; Neamtu and Schumaker 2004). Some work has been done in developing algorithms for approximating solutions of spherical PDE in spherical spline spaces as well (Baramidze and Lai 2006). We continue work in this direction by demonstrating how spherical splines can be employed in a numerical algorithm approximating a solution of the heat equation on the unit sphere.

Consider the problem of heat distribution on the unit sphere. The temperature u at any surface point v and time t on $[0, \infty)$ satisfies the differential equation:

$$\frac{\partial}{\partial t}u(v,t) - \Delta^* u(v,t) = f(v,t), \tag{1}$$

subject to the initial condition

$$u(v,0) = g(v).$$

Here Δ^* is the Laplace–Beltrami operator, the spherical analog of the Laplacian, defined on the unit sphere $\mathbb{S}^2$ by

$$\Delta^* u = [\Delta u_0]|_{\mathbb{S}^2}.$$

The function $u_0(v) = u\left(\frac{v}{|v|}\right)$ is the constant homogeneous extension of u to $\mathbb{R}^3\backslash\{0\}$. We consider and compare performance of two methods for obtaining an approximation of the solution to the equation (1). Section 2 is devoted to the development of the spherical harmonics expansion of the solution. We estimate an L^2 error bound for the approximation of the solution by partial sums. While this development is not new (Orzag 1974), we include it here for completeness and convenience. In Section 3 we discuss the divided difference approximation to the time derivative, and develop an iterative algorithm based on spherical splines. Error estimates are discussed as well. Numerical experiments are described, and the results are summarized in Section 4.

2. Spherical Harmonics

A classical tool available for solving problems on the unit sphere is an orthonormal set of spherical harmonics. One way of approximating a spherical function is expanding it in a series of spherical harmonics and cutting off the tail of the series. Since any L^2 integrable function on the unit sphere has a unique expansion and the series converges, the idea seems very attractive. It is especially appealing in the context of PDE involving the Laplace–Beltrami operator, since the spherical harmonics are the eigenfunctions of the named operator. Even though we are claiming several advantages of spherical splines over spherical harmonics for the problem at hand, the harmonics are very important in the error analysis of the weak solutions (Baramidze and Lai 2006), and in certain situations provide quick and accurate solutions.

Case 1. We begin by considering the heat distribution on the unit sphere with no heat source

$$\frac{\partial}{\partial t}w(v,t) - \Delta^* w(v,t) = 0. \tag{2}$$

We seek a separable solution $w \in C^2(\mathbb{S}^2) \times C^1([0,T])$ of the form

$$w(v,t) = \upsilon(v)\tau(t).$$

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The standard procedure leads to separated equations

$$\frac{\partial}{\partial t}\tau(t) = K\tau(t)$$

and

$$\Delta^* \upsilon(v) = K\upsilon(v)$$

for some separation constant K. The first equation has solutions of the form

$$\tau(t) = Ce^{Kt}$$

for any constant C. Solutions of the second equation are spherical harmonics $Y_{\ell k}$, since they are eigenfunctions of Laplace–Beltrami operator

$$\Delta^* Y_{\ell k} = -\ell(\ell+1)Y_{\ell k}, \ell \in \mathbb{N} \cup \{0\}, k = 0, \cdots, 2\ell.$$

Therefore

$$\upsilon(v) \in \{Y_{\ell k}(v), \ell \in \mathbb{N} \cup \{0\}, k = 0, \cdots, 2\ell\},$$

and the separation constant K takes on values $-\ell(\ell+1)$. We have

$$w(v,t) = CY_{\ell k}(v)e^{-\ell(\ell+1)t}, \ C \in \mathbb{R}, \ \ell \in \mathbb{N}\cup\{0\}, \ k = 0, \cdots, 2\ell.$$

Note that the function $w(v,t) = CY_{\ell k}(v)e^{-\ell(\ell+1)t}$ solves the homogeneous problem (2) for every $C \in \mathbb{R}, \ell \in \mathbb{N} \cup \{0\}, k = 0, \cdots, 2\ell$.

Lemma 1: A general solution of the homogeneous problem (2) is

$$w(v,t) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}Y_{\ell k}(v)e^{-\ell(\ell+1)t}. \tag{3}$$

Proof: The set $\{Y_{\ell k}, \ell \geq 0, k = 0, \cdots, 2\ell\}$ forms an $L^2(\mathbb{S}^2)$-orthonormal system. A classical solution $w(v,t)$ of the problem (2) is continuous and bounded on $\mathbb{S}^2$ and thus has a unique expansion in terms of spherical harmonics for every fixed t, i.e.

$$w(v,t) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}(t)Y_{\ell k}(v),$$

where the coefficients $C_{\ell k}$ are square summable. Since $w(v,t)$ is differentiable with respect to t

$$\frac{\partial}{\partial t}w(v,t) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} \frac{\partial}{\partial t}C_{\ell k}(t)Y_{\ell k}(v).$$

Since $\Delta^* w(v,t)$ is defined for $w(v,t)$ we have

$$\begin{aligned}\Delta^* w(v,t) &= \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}(t)\Delta^* Y_{\ell k}(v) \\ &= -\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} \ell(\ell+1)C_{\ell k}(t)Y_{\ell k}(v).\end{aligned}$$

Therefore (2) becomes

$$\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell}(\frac{\partial}{\partial t}C_{\ell k}(t) + \ell(\ell+1)C_{\ell k}(t))Y_{\ell k}(v) = 0.$$

By the linear independence of spherical harmonics

$$\frac{\partial}{\partial t}C_{\ell k}(t) = -\ell(\ell+1)C_{\ell k}(t)$$

for every ℓ, k. Therefore

$$C_{\ell k}(t) = C_{\ell k}e^{-\ell(\ell+1)t}$$

and

$$w(v,t) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}Y_{\ell k}(v)e^{-\ell(\ell+1)t}$$

is a solution of (2).

Case 2. Consider now a problem with a time-independent heat source

$$\frac{\partial}{\partial t}u(v,t) - \Delta^* u(v,t) = f(v), \tag{4}$$

subject to the initial condition

$$u(v,0) = g(v). \tag{5}$$

Differentiate (4) with respect to time and denote $w = \frac{\partial}{\partial t}u$. We obtain (2) which has solutions of the form (3). Then we must have for $u(v,t)$

$$u(v,t) = \int \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}Y_{\ell k}(v)e^{-\ell(\ell+1)t}dt =$$

$$C_{00}Y_{00}t + \sum_{\ell=1}^{\infty}\sum_{k=-\ell}^{\ell} \frac{C_{\ell k}}{-\ell(\ell+1)}Y_{\ell k}(v)e^{-\ell(\ell+1)t} + K(v)$$

for some function $K(v)$ independent of time. The initial condition (5) implies

$$\sum_{\ell=1}^{\infty}\sum_{k=-\ell}^{\ell} \frac{C_{\ell k}}{-\ell(\ell+1)}Y_{\ell k}(v) + K(v) = g(v),$$

and therefore we obtain

$$\begin{aligned}u(v,t) &= C_{00}Y_{00}t \\ &+ \sum_{\ell=1}^{\infty}\sum_{k=-\ell}^{\ell} \frac{C_{\ell k}}{\ell(\ell+1)}Y_{\ell k}(v)(1-e^{-\ell(\ell+1)t}) \\ &+ g(v),\end{aligned}$$

where the coefficients $C_{\ell k}$ are still to be determined. Computing time derivative of $u(v,t)$ and $\Delta^* u$

$$\frac{\partial}{\partial t}u(v,t) = C_{00}Y_{00} + \sum_{\ell=1}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}Y_{\ell k}(v)e^{-\ell(\ell+1)t},$$

$$\begin{aligned}\Delta^* u(v,t) &= -\sum_{\ell=1}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k} Y_{\ell k}(v)(1-e^{-\ell(\ell+1)t}) \\ &+ \Delta^* g(v),\end{aligned}$$

we get in (4)

$$f(v) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k} Y_{\ell k}(v) - \Delta^* g(v).$$

Theorem 1: The time-independent heat source equation (4) subject to the initial condition (5) with $f + \Delta^* g \in L^2(\mathbb{S}^2)$ has a unique solution

$$\begin{aligned}u(v,t) &= C_{00} Y_{00} t \\ &+ \sum_{\ell=1}^{\infty}\sum_{k=-\ell}^{\ell} \frac{C_{\ell k}}{\ell(\ell+1)} Y_{\ell k}(v)(1-e^{-\ell(\ell+1)t}) \\ &+ g(v), \end{aligned} \tag{6}$$

with the coefficients $C_{\ell k}$ subject to

$$\sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k} Y_{\ell k}(v) = f + \Delta^* g. \tag{7}$$

Proof: Considering the above, we only need to show uniqueness of the solution. If u_1 and u_2 both solve (4) subject to (5) their difference $w = u_1 - u_2$ solves the homogeneous equation (2) subject to the zero initial conditions. By (3) and the linear independence of spherical harmonics $w = 0$, and thus $u_1 = u_2$.

Note that the coefficients $C_{\ell k}$ can be found by

$$C_{\ell k} = \int_{\mathbb{S}^2} (f + \Delta^* g) Y_{\ell k} ds.$$

Corollary 1: If $f + \Delta^* g = K$ for a real number K, then

$$u(v,t) = (f(v) + \Delta^* g(v))t + g(v).$$

If $f + \Delta^* g$ has an infinite expansion (7), one can use a finite sum to approximate $f + \Delta^* g$. Let $\epsilon(N)$ denote the L^2 error in the approximation of $f + \Delta^* g$ by the finite linear combination of spherical harmonics $Y_{\ell k}$ with $0 \le \ell \le N$. Then the N-th partial sum u_N calculated according to (6) approximates the exact solution $u(v,t)$ of (4) with

$$\begin{aligned}&\|u - u_N\|_{L^2}^2 = \\ &= \| \sum_{\ell=N+1}^{\infty}\sum_{k=-\ell}^{\ell} \frac{C_{\ell k}(1-e^{-\ell(\ell+1)t})}{\ell(\ell+1)} Y_{\ell k}\|_{L^2}^2 \\ &= \sum_{\ell=N+1}^{\infty}\sum_{k=-\ell}^{\ell} \frac{C_{\ell k}^2 (1-e^{-\ell(\ell+1)t})^2}{\ell^2(\ell+1)^2} \\ &\le \sum_{\ell=N+1}^{\infty}\sum_{k=-\ell}^{\ell} \frac{C_{\ell k}^2}{(N+1)^2(N+2)^2} \\ &\le \frac{1}{N^4}\sum_{\ell=N+1}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}^2 = \frac{\epsilon(N)^2}{N^4}.\end{aligned}$$

Corollary 2: Suppose $f + \Delta^* g \in L^2(\mathbb{S}^2)$ is approximated by the N-th partial sum with the L^2 error $\epsilon(N)$, then the solution u and its partial sum approximation u_N satisfy

$$\|u - u_N\|_{L^2} \le \frac{\epsilon(N)}{N^2}.$$

Case 3. In case of a time-dependent heat source we attempt to find a solution of the form

$$u(v,t) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}(t) Y_{\ell k}(v).$$

Substituting u into the equation and using the expansion of f as

$$f(v,t) = \sum_{\ell=0}^{\infty}\sum_{k=-\ell}^{\ell} f_{\ell k}(t) Y_{\ell k}(v),$$

we get for the coefficients $C_{\ell k}(t)$ of u

$$\frac{\partial}{\partial t} C_{\ell k}(t) + \ell(\ell+1) C_{\ell k}(t) = f_{\ell k}(t).$$

The nonhomogeneous linear first order ordinary differential equation above has a solution

$$C_{\ell k}(t) = e^{-\ell(\ell+1)t}\left(\int_0^t e^{\ell(\ell+1)\tau} f_{\ell k}(\tau) d\tau + g_{\ell k}\right),$$

where $g_{\ell k}$ are the coefficients of the initial value function $u(v,0) = g(v)$ in the series of spherical harmonics. Recalling that

$$f_{\ell k}(t) = \int_{\mathbb{S}^2} f(v,t) Y_{\ell k} ds,$$

we finally get

Theorem 2: The coefficients in the spherical harmonics expansion of the solution u of (1) are defined by

$$\begin{aligned}&C_{\ell k}(t) = e^{-\ell(\ell+1)t} \times \\ &\left(\int_{\mathbb{S}^2}\left(\int_0^t e^{\ell(\ell+1)\tau} f(v,\tau) d\tau\right) Y_{\ell k} ds + g_{\ell k}\right)\end{aligned} \tag{8}$$

To estimate the error in the approximation of u by its N-th partial sum u_N at some fixed moment of time t, we first notice that, by the triangle inequality

$$\begin{aligned}\|u(t) - u_N(t)\|_{L^2} &= \left(\sum_{\ell=N+1}^{\infty}\sum_{k=-\ell}^{\ell} C_{\ell k}^2(t)\right)^{1/2} \\ &\le \left(\sum_{\ell=N+1}^{\infty}\sum_{k=-\ell}^{\ell} (\int_0^t e^{\ell(\ell+1)(\tau-t)} f_{\ell k}(\tau) d\tau)^2\right)^{1/2} \\ &+ \left(\sum_{\ell=N+1}^{\infty}\sum_{k=-\ell}^{\ell} g_{\ell k}^2\right)^{1/2}.\end{aligned}$$

Consider the first term on the right. By Cauchy-Schwarz inequality

$$\begin{aligned}&(\int_0^t e^{\ell(\ell+1)(\tau-t)} f_{\ell k}(\tau) d\tau)^2 \\ \leq & \int_0^t e^{2\ell(\ell+1)(\tau-t)} d\tau \int_0^t f_{\ell k}^2(\tau) d\tau.\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{\ell=N+1}^{\infty} \sum_{k=-\ell}^{\ell} (\int_0^t e^{\ell(\ell+1)(\tau-t)} f_{\ell k}(\tau) d\tau)^2 \\ \leq & \sum_{\ell=N+1}^{\infty} \sum_{k=-\ell}^{\ell} \frac{1}{2\ell(\ell+1)} \int_0^t f_{\ell k}^2(\tau) d\tau \\ \leq & \frac{1}{2N^2} \int_0^t \left(\sum_{\ell=N+1}^{\infty} \sum_{k=-\ell}^{\ell} f_{\ell k}^2(\tau) \right) d\tau \\ \leq & \frac{1}{2N^2} \int_0^t \left(\|f(\cdot,\tau) - f_N(\cdot,\tau)\|_{L^2}^2 \right) d\tau \\ \leq & \frac{t}{2N^2} \left(\max_{\tau\in[0,t]} \|f(\cdot,\tau) - f_N(\cdot,\tau)\|_{L^2} \right)^2.\end{aligned}$$

We thus have the following

Corollary 3:

$$\begin{aligned}&\|u-u_N\|_{L^2} \leq \\ \leq & \frac{\sqrt{t/2}}{N} \left(\max_{\tau\in[0,t]} \|f(\cdot,\tau) - f_N(\cdot,\tau)\|_{L^2} \right) \\ + & \|g-g_N\|_{L^2}, \end{aligned} \tag{9}$$

where f_N and g_N denote the N-th partial sum approximations of f and g respectively.

It is not difficult to see that

$$\|g-g_N\|_{L^2} \leq \frac{1}{N^s} \|g\|_{H^s},$$

for an L^2 integrable function in the Sobolev space H^s equipped with the norm

$$\|g\|_{H^s}^2 := \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} (1+\ell(\ell+1))^s |g_{\ell k}|^2.$$

Therefore (9) implies rapid convergence for smooth spherical functions.

We conclude this section with a recursive definition of an orthonormal set of spherical harmonics as trivariate functions in $\mathbb{R}^3$ based on the three-term recursion formula for Legendre polynomials $p_\ell(z)$.

Initialize

$$\begin{aligned} p_0 &= 1, \\ p_1 &= z, \\ s_1 &= y, \\ c_1 &= x. \end{aligned}$$

Define

$$\begin{aligned} Y_{\ell,0} &= \sqrt{\frac{2\ell+1}{4\pi}} p_\ell, \ell = 0, 1 \\ Y_{1,-1} &= \sqrt{\frac{3}{4\pi}} s_1, \\ Y_{1,1} &= \sqrt{\frac{3}{4\pi}} c_1. \end{aligned}$$

For $\ell \geq 2$ define

$$\begin{aligned} p_\ell &= \frac{1}{\ell}(p_{\ell-1} z(2\ell-1) - (\ell-1)p_{\ell-2}) \\ s_\ell &= s_1 c_{\ell-1} - s_{\ell-1} c_1 \\ c_\ell &= c_1 c_{\ell-1} + s_1 s_{\ell-1} \\ Y_{\ell,0} &= \sqrt{\frac{2\ell+1}{4\pi}} p_\ell. \end{aligned}$$

For every such ℓ and $k = 1, \cdots, \ell$ define

$$\begin{aligned} Y_{\ell,-k} &= \sqrt{\frac{(2\ell+1)(\ell-k)!}{2\pi(\ell+k)!}} s_k \frac{d}{dz^k} p_\ell, \\ Y_{\ell,k} &= \sqrt{\frac{(2\ell+1)(\ell-k)!}{2\pi(\ell+k)!}} c_k \frac{d}{dz^k} p_\ell. \end{aligned}$$

3. Spherical Splines

In (Baramidze and Lai, 2006) we have shown how to use spherical splines to solve equations of the form

$$-\Delta^* u + \omega^2 u = F \tag{10}$$

on the unit sphere. We need the following lemma.

Lemma 2: Let $\mathbf{u}$ be the weak solution of (10) in $H^1(\mathbb{S}^2)$ and let $\bar{\mathbf{u}}$ be the weak solution of

$$-\Delta^* u + \omega^2 u = \bar{F}.$$

Then

$$\|\mathbf{u} - \bar{\mathbf{u}}\|_{L^2} \leq \frac{1}{\omega^2} \|F - \bar{F}\|_{L^2}.$$

Proof: Since

$$-\langle \Delta^* \mathbf{u}, \upsilon \rangle + \omega^2 \langle \mathbf{u}, \upsilon \rangle = \langle F, \upsilon \rangle$$

and

$$-\langle \Delta^* \bar{\mathbf{u}}, \upsilon \rangle + \omega^2 \langle \bar{\mathbf{u}}, \upsilon \rangle = \langle \bar{F}, \upsilon \rangle$$

for every test function $\upsilon \in H^1(\mathbb{S}^2)$, the difference of the last two equations yelds

$$-\langle \Delta^* (\mathbf{u} - \bar{\mathbf{u}}), \upsilon \rangle + \omega^2 \langle \mathbf{u} - \bar{\mathbf{u}}, \upsilon \rangle = \langle F - \bar{F}, \upsilon \rangle.$$

Since $\mathbf{u} - \bar{\mathbf{u}} \in H^1(\mathbb{S}^2)$ we get in particular

$$-\langle \Delta^* (\mathbf{u}-\bar{\mathbf{u}}), \mathbf{u}-\bar{\mathbf{u}} \rangle + \omega^2 \langle \mathbf{u}-\bar{\mathbf{u}}, \mathbf{u}-\bar{\mathbf{u}} \rangle = \langle F-\bar{F}, \mathbf{u}-\bar{\mathbf{u}} \rangle.$$

By equation (5) in (Baramidze and Lai, 2006)

$$\langle (-\Delta^* + \omega^2)\upsilon, \upsilon \rangle = \|\upsilon\|_{H^1}^2 + (\omega^2 - 1)\|\upsilon\|_{L^2}^2$$

for any $\upsilon \in H^1$, and therefore for $\upsilon = \mathbf{u} - \bar{\mathbf{u}}$

$$\|\upsilon\|_{H^1}^2 + (\omega^2 - 1)\|\upsilon\|_{L^2}^2 = \langle F - \bar{F}, \upsilon \rangle.$$

By Lemma 2 (Baramidze and Lai, 2006)

$$\|v\|_{L^2} \le \|v\|_{H^1}.$$

Thus

$$\|v\|_{L^2}^2 + (\omega^2 - 1)\|v\|_{L^2}^2 = \omega^2\|v\|_{L^2}^2 \le \langle F - \bar{F}, v\rangle \le \|F - \bar{F}\|_{L^2}\|v\|_{L^2}.$$

The equation (10) is the result of a time domain discretization in (1) by a divided difference. We employ the following approximation of the time derivative of $u(v,t)$:

$$\frac{\partial u}{\partial t}(v, t_k) = \frac{3/2u_k - 2u_{k-1} + 1/2u_{k-2}}{h} + O(h^2),$$

where $u_k = u(v, t_k)$. Then (1) becomes

$$-\Delta^* u_k + \frac{3}{2h}u_k = f_k + \frac{2}{h}u_{k-1} - \frac{1}{2h}u_{k-2} + O(h^2), \tag{11}$$

where $f_k = f(v, t_k)$. Solving

$$-\Delta^* u_k + \frac{3}{2h}u_k = f_k + \frac{2}{h}u_{k-1} - \frac{1}{2h}u_{k-2}, \tag{12}$$

for $\bar{u}_k, k = 2, ...N$ requires two previous values u_{k-1} and u_{k-2}. Since usually only one initial value $u_0 = u(v,0) = g(v)$ is available, we estimate $u_1 = u(v,h)$ using Taylor's expansion near 0. For notational simplicity we are skipping the spacial variable. Also, let us assume that u, g and f are sufficiently smooth. Then

$$\begin{aligned} u(h) &= u(0) + u_t(0)h + u_{tt}(0)\frac{h^2}{2} + O(h^3) \\ &= g + (f(0) + \Delta^* u(0))h \\ &+ (f_t(0) + \Delta^* u_t(0))\frac{h^2}{2} + O(h^3) \\ &= g + (f(0) + \Delta^* g)h \\ &+ (f_t(0) + \Delta^* f(0) + (\Delta^*)^2 g)\frac{h^2}{2} \\ &+ O(h^3). \end{aligned} \tag{13}$$

Note that using

$$\bar{u}_1 = g + (f(0) + \Delta^* g)h + (f_t(0) + \Delta^* f(0) + (\Delta^*)^2 g)\frac{h^2}{2} \tag{14}$$

in place of u_1 in (11) does not affect the order of the defect at the second step

$$\begin{aligned} d_2(h) &= (-\Delta^* u_2 + \frac{3}{2h}u_2) - (-\Delta^* \bar{u}_2 + \frac{3}{2h}\bar{u}_2) \\ &= \frac{2}{h}(u_1 - \bar{u}_1) + O(h^2) = O(h^2). \end{aligned} \tag{15}$$

For further analysis, we estimate the local L^2 error at the k-th step as follows. Initially, $u(v,0) = g(v)$ and $u_1 = \bar{u}_1 + O(h^3)$ according to (13) and (14). Let $\mathbf{u}_2$ solve weakly (10) with $F = f_2 + \frac{2}{h}u_1 - \frac{1}{2h}g + O(h^2)$. Let $\tilde{\mathbf{u}}_2$ solve (10) with $\tilde{F} = f_2 + \frac{2}{h}\bar{u}_1 - \frac{1}{2h}g$. By Lemma 2 with $\omega^2 = \frac{3}{2h}$, and since

$$F - \tilde{F} = \frac{2}{h}(u_1 - \bar{u}_1) = O(h^2),$$

we get

$$\|\mathbf{u}_2 - \tilde{\mathbf{u}}_2\|_{L^2} \le \frac{2h}{3}\|O(h^2)\|_{L^2} = O(h^3).$$

At the next step let $\mathbf{u}_3$ solve (10) weakly with $F = f_3 + \frac{2}{h}\mathbf{u}_2 - \frac{1}{2h}u_1 + O(h^2)$. Let $\tilde{\mathbf{u}}_3$ solve (10) with $\tilde{F} = f_3 + \frac{2}{h}\tilde{\mathbf{u}}_2 - \frac{1}{2h}\bar{u}_1$. Since

$$F - \tilde{F} = \frac{2}{h}(\mathbf{u}_2 - \tilde{\mathbf{u}}_2) - \frac{1}{2h}(u_1 - \bar{u}_1)$$

we get

$$\|F - \tilde{F}\|_{L^2} = \frac{2}{h}\|\mathbf{u}_2 - \tilde{\mathbf{u}}_2\|_{L^2} + \frac{1}{2h}\|u_1 - \bar{u}_1\|_{L^2} = O(h^2).$$

Therefore

$$\|\mathbf{u}_3 - \tilde{\mathbf{u}}_3\|_{L^2} = O(h^3)$$

as well. It is not too difficult to notice however that a constant in $O(h^2)$ is accumulating proportionally to k. Making an inductive argument we conclude that

Theorem 3: There exists a positive constant c depending on smoothness of the solution u of (1) and k such that

$$\|\mathbf{u}_k - \tilde{\mathbf{u}}_k\|_{L^2} \le ckh^3. \tag{16}$$

If we were to consider the error at the end of a time interval of length T, $k = \frac{T}{h}$ makes the global error to be of order $O(h^2)$.

To approximate the solution of the PDE (12) by spherical splines we first partition the domain. Let Δ be a triangulation of the unit sphere based on vertices $\mathcal{V}$ where function values $f(v, t_k)$ are known at every time step. We assume that the initial condition function values $g(v)$ are given on $\mathcal{V}$ as well. Let $S_d^r(\Delta)$ denote the space of spherical homogeneous Bernstein–Bezier splines of degree d and smoothness r on Δ. Let $\tilde{u}_k$ denote the spline approximation of (12), and $|\Delta|$ denote the size of the largest triangle in Δ (i.e. the diameter of the smallest spherical cap subscribing the largest triangle). By Theorem 4 (Baramidze and Lai, 2006)

$$\|\tilde{\mathbf{u}}_k - \tilde{u}_k\|_{L^2} \le \|\mathbf{u}_k - \tilde{u}_k\|_{H^1} \le C|\Delta|^{m-1},$$

for some $C > 0$ depending on the degree d of the spline space, some triangulation parameters, h, and the smoothness of u. The value of m depends on smoothness of u and is at most d. Therefore for a sufficiently smooth function we expect

$$\|\tilde{\mathbf{u}}_k - \tilde{u}_k\|_{L^2} \le C|\Delta|^{d-1} \tag{17}$$

for all $k \ge 2$. Putting together (16) and (17) we get

Theorem 4: The spherical spline approximation $\tilde{u}_k$ to the weak solution $\mathbf{u}_k$ of the heat equation (1) at the k-th time step satisfies

$$\|\mathbf{u}_k - \tilde{u}_k\|_{L^2} \le C|\Delta|^{d-1} + ckh^3. \tag{18}$$

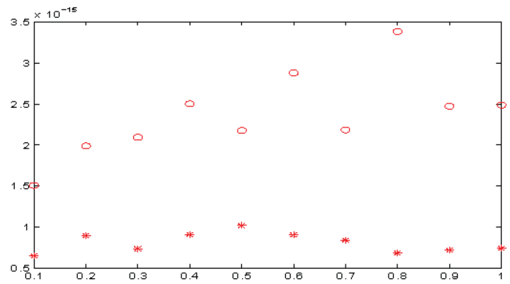

*Figure 1. Reproduction of exact solutions. The error in the approximation of $u = t^2$ is marked by *, and the error in the approximation of $u = xt^2$ is marked by $\circ$*

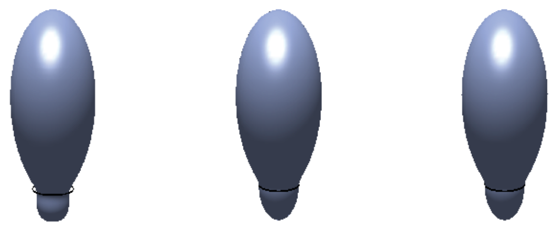

Figure 2. From left to right: spherical harmonic approximation, exact solution, spherical spline approximation.

Table 1. Convergence with respect to the time step size, spline space $S_4^1(\Delta_1)$.

h	$e(h)$	$e(h)/e(h/2)$
0.1	$4.9518e-3$	3.6964
$0.1/2$	$1.3397e-3$	3.8639
$0.1/2^2$	$3.4671e-4$	3.9331
$0.1/2^3$	$8.8152e-5$	3.9638
$0.1/2^4$	$2.2239e-5$	

Table 2. Spherical spline approximation in $S_3^1(\Delta)$, $S_4^1(\Delta)$ and $S_5^1(\Delta)$.

d	3	4	5
Δ_1	$1.8846e-1$	$1.6777e-2$	$2.1826e-2$
Δ_2	$1.0159e-2$	$3.8933e-4$	$2.5366e-4$
Δ_3	$5.0118e-4$	$2.4369e-5$	$3.7372e-6$
Δ_4	$3.4034e-5$	$1.3753e-6$	$6.5130e-8$
Δ_5	$2.9895e-6$	$8.0408e-8$	$1.0727e-9$

Table 3. Convergence of spherical splines in $S_3^1(\Delta)$, $S_4^1(\Delta)$ and $S_5^1(\Delta)$.

d	3	4	5
$\frac{e(\Delta_1)}{e(\Delta_2)}$	18.5510	43.0916	86.0476
$\frac{e(\Delta_2)}{e(\Delta_3)}$	20.2702	15.9769	67.8737
$\frac{e(\Delta_3)}{e(\Delta_4)}$	14.7257	17.7188	57.3800
$\frac{e(\Delta_4)}{e(\Delta_5)}$	11.3853	17.1039	60.7186

Table 4. Error converges as $O(|\Delta|^{d-1})$ when h is reduced by a factor of $2^{(d-1)/2}$.

d	3	4	5
h_1	$6.4782e-1$	$7.0864e-2$	$1.0440e-1$
$\frac{h_1}{2^{(d-1)/2}}$	$4.6735e-2$	$6.2096e-3$	$3.2844e-3$
$\frac{h_1}{2^{d-1}}$	$3.7111e-3$	$7.1917e-4$	$1.8915e-4$
$\frac{h_1}{2^{3(d-1)/2}}$	$8.2075e-4$	$9.2063e-5$	$1.1793e-5$
$\frac{h_1}{2^{2(d-1)}}$	$1.9173e-4$	$1.1637e-5$	$7.3249e-7$

4. Numerical examples

Example 1: In this example we demonstrate reproduction of certain exact solutions. Consider

$$\frac{\partial}{\partial t}u(v,t) - \Delta^* u(v,t) = 2t,$$
$$u(v,0) = 0,$$

with $u(v,t) = t^2$. We expect our algorithm to reproduce u in a space of spherical splines of even degree. Let Δ_1 be a triangulation of the sphere based on the vertices $(\pm1,0,0), (0,\pm1,0), (0,0,\pm1)$ consisting of eight similar triangles. Figure 1 shows $e_k(\Delta_1) = \frac{\|u_k-\tilde{u}_k\|_2}{\|u_k\|_2}$ for 10 steps, $(h = 0.1)$, in $S_4^1(\Delta_1)$. Error values are computed at more than $30,000$ points, with $\max e_k(\Delta_1) = 1.0180e-015$. Similarly, consider

$$\frac{\partial}{\partial t}u(v,t) - \Delta^* u(v,t) = 2x(t+t^2),$$
$$u(v,0) = 0,$$

with $u(v,t) = xt^2$. We expect our algorithm to reproduce u in a space of spherical splines of odd degree. Figure 1 shows $e_k(\Delta_1) = \frac{\|u_k-\tilde{u}_k\|_2}{\|u_k\|_2}$ for 10 steps, $(h = 0.1)$, in $S_5^1(\Delta_1)$. Error values are computed over approximately $30,000$ points, with $\max e_k(\Delta_1) = 3.3865e-015$.

Example 2: The solution of

$$\frac{\partial}{\partial t}u(v,t) - \Delta^* u(v,t) = -e^{-t}$$
$$u(v,0) = 1,$$

is $u(v,t) = e^{-t}$. Note that both $f(v,t) = -e^{-t}$ and $g(v) = 1$ are constant functions on the unit sphere with respect to the spacial variables. Therefore the error in approximation of u_k by an even degree spline $\tilde{u}_k$ is due to the approximation of $\frac{\partial u}{\partial t}$ by the backward difference. We work with the initial triangulation Δ_1 and splines of degree 4 and smoothness 1. We begin with the step size $h = 0.1$ and find the spline approximation to the exact solution on the interval $[0,1]$. We calculate the relative error $e_k(\Delta_1) = \frac{\|u_k-\tilde{u}_k\|_2}{\|u_k\|_2}$ evaluated over more than $30,000$ locations almost uniformly spread on the unit sphere. In Table 1 we record $e(h)$, the maximal value of $e_k(\Delta_1), k = 0,\cdots,10$. We reduce the step size in half several times and record the results in the second column of Table 1. Finally, to illustrate the order of convergence, we calculate the ratios of error of the form $\frac{e(h)}{e(h/2)}$ for all $h = 0.1,\cdots,0.1/2^3$ and record them in the last column of Table 1. Since the ratios approach 4, the convergence of the spline solution to the exact solution is bounded by h^2.

Table 5. Ratios of error converge as 2^{d-1} when h is reduced by a factor of $2^{(d-1)/2}$.

d	3	4	5
$\frac{\|u-\tilde{u}\|_{\infty,\Delta_1,h_1}}{\|u-\tilde{u}\|_{\infty,\Delta_2,h_2}}$	13.8616	11.4119	31.7856
$\frac{\|u-\tilde{u}\|_{\infty,\Delta_2,h_2}}{\|u-\tilde{u}\|_{\infty,\Delta_3,h_3}}$	12.5933	8.6344	17.3641
$\frac{\|u-\tilde{u}\|_{\infty,\Delta_3,h_3}}{\|u-\tilde{u}\|_{\infty,\Delta_4,h_4}}$	4.5216	7.8117	16.0387
$\frac{\|u-\tilde{u}\|_{\infty,\Delta_4,h_4}}{\|u-\tilde{u}\|_{\infty,\Delta_5,h_5}}$	4.2808	7.9111	16.0998

Table 6. N-th partial sum approximation.

N	$\max_{t\in[0,1]} \frac{\|u-u_N\|_2}{\|u\|_2}$	Time, sec.
1	$1.2671e-1$	543
2	$2.0353e-2$	1211
3	$2.5106e-3$	2363
4	$2.5085e-4$	4557
5	$2.1191e-5$	6120
6	$1.5026e-6$	8825
7	$9.3016e-8$	12481
8	$5.1358e-9$	17578
9	$2.7098e-10$	23715
10	$1.7850e-10$	30583

Table 7. N-th partial sum approximation with least squares solution.

N	$\max_{t\in[0,1]} \frac{\|u-u_N\|_2}{\|u\|_2}$
1	$1.2690e-1$
2	$2.1505e-2$
3	$4.0609e-3$
4	$3.0063e-4$
5	$1.2150e-4$

Table 8. Spherical splines versus spherical harmonics.

Time, t_k	Harmonics, $\frac{\|u_k-u_{5,k}\|_2}{\|u_k\|_2}$ $\times e-3$	Splines, $\frac{\|u_k-\tilde{u}_k\|_2}{\|u_k\|_2}$ $\times e-3$
0.1	8.40	2.35
0.2	11.7	0.74
0.3	15.2	0.48
0.4	18.8	0.36
0.5	22.1	0.29
0.6	26.1	0.28
0.7	29.9	0.20
0.8	33.9	0.18
0.9	40.0	0.16
1.0	44.1	0.14

Example 3: In this example we consider the problem

$$\frac{\partial}{\partial t}u(v,t) - \Delta^* u(v,t) = e^x(1+t(2x-1+x^2))$$
$$u(v,0)=0.$$

The exact solution is $u(v,t) = te^x$. Note that since u is linear in time, the error in the approximation of u by the spline $\tilde{u}$ is due to the spline approximation only. We expect the error to behave as $|\Delta|^{d-1}$. Let Δ_1 be a triangulation of the unit sphere as in Example 1. We obtain triangulations Δ_i from Δ_{i-1} by bisecting the edges of triangles and connecting the midpoints. We apply the computational algorithm in $S^1_d(\Delta_i), i = 1,2,3,4,5,\ d = 3,4,5$ on the time interval $[0,1]$ with $h = 0.1$. The relative errors of the form $\max_k\{\frac{\|u_k-\tilde{u}_k\|_2}{\|u_k\|_2}\}$ are recorded in Table 2. The ratios of error as we refine triangulations are recorded in Table 3.

The ratios of error are much higher than the expected values of $4,\ 8$ and 16, however our next example demonstrates that our error estimates are correct.

Example 4: According to (18) to have the overall error of order h^2, we can keep $|\Delta|^{d-1}$ proportional to h^2. And alternatively to have the error of order $|\Delta|^{d-1}$ we must keep h^2 proportional to $|\Delta|^{d-1}$. In general we would like to achieve

$$\|\mathbf{u}_k - \tilde{u}_k\|_{L^2} \le C|\Delta|^{\max\{d-1,2\}},$$

or

$$\|\mathbf{u}_k - \tilde{u}_k\|_{L^2} \le Ch^{\max\{d-1,2\}}.$$

If the size of the triangulation Δ_i is roughly halved, as we have done in the previous examples, the time step size h has to be reduced by a factor of $2^{\frac{d-1}{2}}$.

The function $u(v,t) = e^x \sin t$ solves the problem

$$\frac{\partial}{\partial t}u(v,t) - \Delta^* u(v,t) = e^x(\cos t + \sin t(x^2+2x-1))$$
$$u(v,0)=0.$$

As in the previous example we find the spline approximation of u on four triangulations $\Delta_i, i = 1,\cdots,5$. We use the spline spaces $S^1_d(\Delta_i), d = 3,4,5$ with time step $h_1 = \pi/10$ reduced as shown in the first column of Table 4, and register the maximal absolute error evaluated over $30,000$ locations. Table 4 contains the absolute errors, $\|u-\tilde{u}\|_{\infty,\Delta_i,h_i}, i = 1,\cdots,5$ with respect to supremum norm, for each degree. Table 5 contains the corresponding ratios of error.

Example 5: We run several numerical experiments with spherical harmonics. Consider the problem in Example 3. We calculate the coefficients $C_{\ell k}(t)$ according to (8) for $\ell \le N$ with $N = 1,2,\cdots,10$. We use time step $h = 0.1$ to find an approximation to the solution in the time interval $[0,1]$. For each time step we calculate an ℓ^2 norm of the error over 30,000 points on the unit sphere. The maximal relative error values over $[0,1]$ are recorded in Table 6 together with the time it takes the program to calculate the approximation. The time increases cubically with N, and the error decays by a factor of ≈ 10 with every increment of N (until it reaches the level of integration tolerance).

In this example we assume that the functions f and g are defined everywhere on the sphere, that is the integrals for $f_{\ell k}$'s can be approximated very well. Naturally, numerical integration algorithms have to be adaptive since as the degree of harmonics increases their oscillation increases as well. The coefficients $f_{\ell k}$ and $g_{\ell k}$ are found using Matlab's numerical quadratures with tolerance 10^{-10}. Spherical harmonics provide us with accurate solutions in a resonable time.

Now we consider another scenario, similar to the one when we use spherical splines. Suppose the functions f and g are only known at the discrete set of values, and if this set is not very large then numerical integration for the coefficients of harmonics cannot be very accurate. The alternative is then to solve a system of linear equations of the form $\sum_{\ell,k} g_{\ell k} Y_{\ell k}(v_i) = g(v_i), i = 1, \cdots, M$. The least squares solution is practical and thus the cardinality of the data set N keeps the highest degree of the harmonics N low ($N << \lfloor \sqrt{M} \rfloor - 1$). This prevents us from arbitrarily improving the accuracy of approximation (as we have in Table 6). Assume also that $f(v_i, t)$ can be sampled at any moment of time, that is the integrals

$$\int_0^t e^{\ell(\ell+1)\tau} f_{\ell,k}(v_i, \tau) d\tau \tag{19}$$

can be evaluated numerically for every fixed v_i, and this approximation does not contribute to the error. In the last triangulation we use in Example 2, the number of data locations is 1026. To keep our experiment similar we sample f and g at the same locations for the spherical harmonics approximation. Past certain value, as the largest degree of the harmonics used increases, the accuracy of the coefficients begins to suffer. Moreover, at some point there is not enough information to determine all of the coefficients, and a further increase of the degree becomes useless. In Table 7 we record error values similar to those in Table 6. As seen in the last rows there is almost no improvement for $N = 5$, and the result is much worse than what we have in Table 6.

Comparing the error to the spline results in Table 2 we notice that for the same number of points (last row) all three spline spaces produce more accurate solutions, and that to achieve the same order of accuracy it is enough to use Δ_3, which is based on 66 vertices, for the splines of degree 4 and 5. Spherical spline approximation is taking considerably longer time to approximate the solution, however it seems the way to go when the data is limited.

Note here also, that the time stepping in our spherical spline scheme does not require function evaluations anywhere else but at the time nodes. For the time integrals in spherical harmonic approximation the numerical integration certainly requires more than just the nodal values. If these are not available the situation gets worse.

Spherical spline technique requires numerical integration as well. However most of these integrals are independent of f and t. It is possible to pre-compute the matrices involved in the solution and use them for any number of functions as long as the triangulation does not change.

Example 6: We conclude this section with an example that involves a spatially discontinuous heat source. Consider the problem

$$\frac{\partial}{\partial t} u(v,t) - \Delta^* u(v,t) = f(v,t)$$
$$u(v,0) = 0,$$

with

$$f(v,t) = \begin{cases} 1, \ z < 0 \\ -8t(1-3z^2) + 4z^2 + 1, \ z \geq 0. \end{cases}$$

The exact solution to the problem is

$$u(v,t) = \begin{cases} t, \ z < 0 \\ t(4z^2+1), \ z \geq 0. \end{cases}$$

We approximate the solution to the problem using both spherical harmonics and spherical splines. The values of f and g are sampled at 1026 locations (as in Example 5). The coefficients for the spherical harmonic expansions of f and g are calculated from the resulting linear system using the least squares method. The highest degree of the harmonics used is 5. We still assume that $f(v_i, t)$ can be sampled at any moment of time, that is the integrals (19) can be evaluated numerically for every fixed v_i, and this approximation does not contribute to the error. The relative error is recorded in Table 8. In the same table we record the relative error in the spherical spline approximation in $S_4^1(\Delta_5)$ at every time step $t_k = hk, k = 1, \cdots, 10, h = 0.1$. Note the drastic difference in the error, which is smaller for the splines sometimes by a factor of hundreds.

Figure 2 depicts the exact solution and the two spherical approximations at $t = 1$. Note that infinitely smooth spherical harmonics handle the discontinuity in the derivative of the solution along the line $z = 0$ differently from splines.

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