Journal of Technical Science and Technologies; ISSN 2298-0032; Volume 3, Issue 2, 2014

Finite Dimensional Linear Operators in the Problems of the Theory of Electrical Circuits

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Abstract

The paper is devoted to substantiation of Intermediate Problem method for multi loop electrical circuits' analysis. All pure-loop and pure-nodeircuits contained in the group GC of the initial primitive circuit possess pairwise equal eigenvalues equal in their turn to the eigenvalues of the primitive circuit. The matrices of impedances and conductances and are reciprocal, i.e. is the resolvent of , and vice versa. The Weinstein function for the LC-circuit described by the loop matrix Z(k) is the determinant of a lower right submatrix of order n-k. The Weinstein function for the LC-circuit described by the node matrix Y(n-k) is the determinant of an upper submatrix of order k of the resolvent Zn lying at the intersection of the first k rows and columns of the matrix Z(n) . The determinant of the conductance node matrix of an arbitrary k-loop LC-circuit is the Weinstein function for the loop impedance matrix of this circuit, and vice versa.

Keywords: constraints transformation matrices, orthogonal, pure-loop, pure-node, primitive circuits, projection operator, resolvent

Introduction

Weinstein's method of intermediate problems was developed for infinite-dimensional problems, for which it proved to be sufficiently effective, especially for problems connected with oscillations of membranes of various configurations (Weinstein A.1971, Aronszain N.1950). However, the part of the method that was developed for finite-dimensional problems had no practical importance. The reason can be explained as follows: the application of Weinstein's function and especially of Aronszajn's lemma require that the resolvent R_λ be calculated for each tested value, which is absolutely impossible to do in the case of large scale (many-loop) circuits. Recall that the matrix R_λ is the inverse matrix to the impedances matrix Z^n. Another point, which is probably the main one, is that if in the infinite-dimensional case we manage to construct a series of intermediate problems (by changing the boundary conditions, which is equivalent to imposing successively constraints) converging to the initial one and to estimate the eigenvalues from below (the classical Rietz and Galerkin methods, collocation method and others give estimates from above), while in the finite-dimensional case. It is not clear how to use Aronszajn's lemma in general, since the method of intermediate problems does not give any clues as to how one can construct the so-called basic problem. i.e. how to determine the basic n-dimensional operator Z^n.

All the mentioned problems associated with Weinstein's function and Aronszajn's lemma become solvable in next to no time and lead to new significant results if this method is modified so as to conform to the notions of primitive and pure-loop (pure-node) circuits (Kron G., 1959, A. Milnikov, A. Prangishvili 2014, Mylnikov A. 2008)

We used not conventional conceptions and terms as: pure-loop, pure-node, orthogonal, primitive circuits, their groups etc. Detailed explanation can be found in (Kron G., 1959, A.Milnikov, A. Prangishvili 2014,Mylnikov A. 2008)

Foundation of Intermediate problems for multi loop circuits

Let us transform the initial k-loop circuit to a pure-loop circuit by shorting n-k node pairs. To this circuit there corresponds the n-dimensional operator Z(n). Moreover, we have a primitive circuit, to which there corresponds also an n-dimensional operator ZD (Kron G., 1959. A. Milnikov, A. Prangishvili 2014)

The matrix of this operator is diagonal:

$$Z_D = \begin{vmatrix} \lambda l_{11} - 1/c^{11} & & & \\ & \lambda l_{22} - 1/c^{22} & & \\ & & \dots & \\ & & & \lambda l_{nn} - 1/c^{nn} \end{vmatrix} \quad (1)$$

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Journal of Technical Science and Technologies; ISSN 2298-0032; Volume 3, Issue 2, 2014

The matrices Z^n and Z_D are related through

$Z^{(n)} = C^T ZDC$, where C is an n ×n matrix of transformation of the initial primitive circuit to the connected pure-loop one.

The inverse matrix to (1) is the resolvent $R_\lambda^{(0)}$ for a primitive circuit written as

$$R_\lambda^0 = \begin{vmatrix} \frac{1}{(\lambda l_{11} - 1/c^{11})} & & & \\ & \frac{1}{\lambda l_{22} - 1/c^{22}} & & \\ & & \cdots & \\ & & & \frac{1}{\lambda l_{nn} - 1/c^{nn}} \end{vmatrix} \tag{2}$$

Using (2) and the matrix C it is easy to obtain the resolvent $R_\lambda^{(n)}$ for the operator $Z^{(n)}$

$$R_\lambda^{(n)} = (Z^{(n)})^{-1} = C^{-1} R_\lambda^{(0)} (C^{-1})^T \tag{3}$$

Relation (3) allows us to reduce the calculation of the resolvent for a pure-loop circuit to a simple matrix calculation: C consists of zeros, -1 and +1; the inverse matrix C^{-1} is easily calculated only once, and all next operations reduce to taking linear combinations of elements $\lambda l_{ii} - 1/c^i$ (i = 1, 2,…, n). We would like to emphasize the fact that (3) is in fact the resolvent obtained in a general form so that we need not to calculate it anew (i.e. to transform the matrix) for each tested λ.

The eigenvalues of the diagonal operator Z_D are obviously equal to

$$\lambda_i = \frac{1}{l_{ii} c_{ii}} \cdot \text{ (i=1,2,...,n)}$$

Important remark: among them there may be multiple eigenvalues too, which from the engineering standpoint means that among the elements used to construct a k-loop circuit there are groups of elements that haveing the same impedances and the quantities of these groups are equal to the corresponding multiplicities.

Now we can make the following statement.

Methodology

Proposition 1.
All pure-loop circuits contained in the group GC of the initial primitive circuit possess pairwise equal eigenvalues equal in their turn to the eigenvalues of the primitive circuit.

Proof. From (3) it follows that

$$\det(R_\lambda^{(n)}) = \det(C' Z_D C)^{-1} = \det(C^{-1}) \det(R_\lambda^{(n)}) \det(C^{-1})^T$$

But det C^T and det C are constant values and therefore the respective determinants are equal to zero only for equal λ

$$\det(R_\lambda^{(n)}) = \det(R_\lambda^{(0)}) = 0 .$$

Analogously, for two arbitrary pure-loop circuits, each of which is obtained by means of a nonsingular transformation C from a given primitive circuit, we can write

$$\det(Z_i^{(n)}) = \det(C_j^T Z_j^{(n)} C_j)$$

where $Z_i^{(n)}$ - the impedance tensor of the i-th pure-loop circuit;

$Z_i^{(n)}$ - the impedance tensor of the j-th pure-loop circuit;

Cij - the tensor of transformation of the basis of the i-th pure-loop circuit to that of the j-th circuit.

From the latter it follows that the equality

$$\det(Z_j^{(n)}) = \det(Z_i^{(n)}) = \det(Z_D)$$

is again full filled for equal λ. Q.E.D

Thus we have obtained two important results:

– the resolvent $R_\lambda^{(n)}$ of the operator $Z^{(n)}$, of a pure-loop circuit can be obtained directly from (3) without transforming the matrix of Z^n;

– the eigenvalues of the initial primitive system are equal to the eigenvalues of a pure-loop circuit or, in other words, the Eigenfrequencies of individual elements, by which the circuit is constructed, are equal to the eigenvalues of the constructed circuit where n-k node pairs are shorted.

The above reasoning has been carried out using the terms of the method of loop currents. The same can also be done in terms of node voltages. In that case, the admittance tensor of a primitive node circuit is the diagonal conductance (admittance) matrix of non connected branches

$$Y_D = \begin{vmatrix} \frac{1}{\lambda l_{11} - 1/c_{11}} & & & \\ & \frac{1}{\lambda l_{22} - 1/c_{22}} & & \\ & & \cdots & \\ & & & \frac{1}{\lambda l_{nn} - 1/c_{nn}} \end{vmatrix}$$

while its resolvent is obviously is

$$R_{\lambda}^{(0)} = \begin{vmatrix} \lambda l_1 - 1/c_1 & & & \\ & \lambda l_2 - 1/c_2 & & \\ & & \dots & \\ & & & \lambda l_m - 1/c_n \end{vmatrix}$$

At that, the eigenvalues are equal to $\lambda_i = l_i c_i$, i.e. they are inverse with respect to the eigenvalues of the pure-loop circuit. And, finally, we paraphrase without proving the statement dual to Proposition 1.

Proposition 2
All pure-node circuits contained in the group GC of the initial primitive circuit possess pairwise equal eigenvalues which are in their turn equal to the eigenvalues of the primitive circuit.

Proposition 1 and 2 imply that in the method of intermediate problems we should consider as a basic problem either a pure-loop circuit or a pure-node circuit because the resolvents of these problems are easily defined in a general form, and the eigenvalues are likewise easily calculated.

As the corresponding basis operator we should consider the impedance tensor Z^n of a pure-loop circuit and the admittance tensor $Y^{(n)}$ in the case of a pure-node circuit.

The next step is to define the concrete form of the intermediate operators, i.e. of impedance tensors $Z^{(n)}$, $Z^{(n-1)}$,..., $Z^{(n-i)}$,…, $Z^{(k)}$.

We proceed from the fact that the operator $Z^{(k)}$ can be obtained from the operator $Z^{(n)}$ by opening successively or at once n-k short circuited node pairs of a pure-loop circuit, which is equivalent to imposing n-k constraints $Z^{(k)} = Z^{n-(n-k)}$.

Let us number all n loops so that fictitious n-k loops would get the last n-k numbers. The opening of the j-th loop obviously leads to the constraint equation

$$i^j = 0$$

To this equation there corresponds the constraint vector pj with components (0… 1… 0), where 1 is in the j-th position.

Thus the k-loop circuit is obtained from the corresponding pure-loop circuit by imposing successively (or simultaneously) n-k constraints to which there correspond n-k mutually orthogonal, unit basis constraint vectors pi (i=1,…,n-k). From the geometric standpoint, the process of imposing r constraints corresponds to the transformation of the operator Z(n) to its part, the latter part being the operator Z(n-r) which is defined on the subspace Ln-r of the space Ln . Let us establish the form of the operator Z(n-r). With this aim in view, we first have to define the coordinate form of the projection operator P.

Proposition 3.
The projection operator P defined for r constraints which are orthonormalized unit vectors has the coordinate form

$$\begin{vmatrix} E & 0_1 \\ 0_2 & 0_3 \end{vmatrix}$$

where E is the (n-r) ×(n-r) identity matrix; 01is the r×r zero matrix; 01 and 02 are the (n-r)×r and r×(n-r) zero matrices.

Proof. Let q be a vector of independent coordinates. Then imposing of h constraints can be expressed as

$$P^h q = q - \sum_{i=1}^{h} (q, p_i) p_i$$

From this definition it obviously follows that for any vector q from L^n the vector $P^h q$ is an orthogonal projection onto the constraint plane.

Substituting the vectors pi (i = n-r-1, n-r-2, …, n) into equality (5.8), we obtain the following coordinate form for the operator P:

$$P^r \begin{vmatrix} q^1 \\ q^2 \\ , \\ , \\ , \\ q^{n-1} \\ q^n \end{vmatrix} = \begin{vmatrix} q^1 & & & & & & \\ & q^2 & & & & & \\ & & \dots & & & & \\ & & & q^{n-r} & & & \\ & & & & 0 & & \\ & & & & & \dots & \\ & & & & & & 0 \end{vmatrix}$$

but an analogous equality may hold only if the vector q $(q^1,…,q^n)$ is subjected to the action of the matrix (5), Q.E.D. ¤

Now we can obtain the concrete representation of the part of the operator Z(n).

Proposition 4.
The operator $Z^{(n-r)}$ which is a part of the operator $Z^{(n)}$ and defined on the subspace Ln-r is represented in the coordinate from as a principal submatrix of order n-r of the matrix $Z^{(n)}$.

Proof. Let us introduce the block notation: for an arbitrary vector q in an n-dimensional space we denote the composite vector by (q_{n-r}, q_r) , where qn-r is a vector of dimension n-r defined in the subspace L^{n-r} , qr is a vector of dimension r defined in the subspace L^r. For an arbitrary matrix Z of order n we denote the block matrix by

$$\begin{vmatrix} Z^{(n-r)} & Z^{n-r,r} \\ Z^{r,n-r} & Z^{(r)} \end{vmatrix},$$

where one upper index denotes order of a square matrix, while two upper indice denote a nonsquare matrix with number of rows equals to the first index, and number of columns - to the second one.

Since the coordinate representation of the operator Z is given in the form of a square impedance matrix Z of order n, for the sake of generality we will carry out the proof for an arbitrary square matrix.

Using the projection operator (5), we can write

$$PZq = \begin{vmatrix} E & 0_2 \\ 0_1 & 0_3 \end{vmatrix} \times \begin{vmatrix} Z^{(n-r)} & Z^{n-r,r} \\ Z^{r,n-r} & Z^{(r)} \end{vmatrix} \times \begin{vmatrix} q_{n-r} \\ q_r \end{vmatrix}$$

whence by performing the operation of multiplication for block matrices we obtain

$$PZq = \begin{vmatrix} Z^{(n-r)} & Z^{n-r,r} \\ 0 & 0 \end{vmatrix} \times \begin{vmatrix} q_{n-r} \\ q_r \end{vmatrix} = \left| Z^{(n-r)} q_{n-r} + Z^{n-r,r} q_r \right|.$$

Taking into account that after imposing r constraints qr= 0 (zero vector), we eventually have

$$PZq=Z^{n-r}q_{n-r}$$

Thus, when constraints of type (4) are successively imposed on pure-node circuit of constraints (which is equivalent to a successive opening of “fictitious” r loops), we obtain a number of intermediatory problems on eigenvalues for a chain of operators

$$Z^{(n)}, Z^{(n-1)},\dots, Z^{(n-i)}, Z^{(n-(i+1))},\dots Z^{(k)}, \qquad (7)$$

each of which (except the first one $Z^{(n)}$) is in coordinate terms a principal submatrix (of order smaller by one) of the preceding operator.

The Weinstein function for oscillatory circuits

Sometimes we'll use notation $Z^{(n)}(\lambda)$ or $Z^{(n)}(\omega^2)$ (in dual case $Y^{(n)}(\lambda)$ or $Y^{(n)}(\omega^2)$) for impedance tensor Z(n) (for conductances tensor Y(n)), thereby emphasizing the dependence of the latter on a frequency value ω^2 . Also we'll use notation $z_i(\lambda)$ or $y_i(\lambda)$ for separate impedances or conductunces of branches with the same emphasis.

The tensor approach has allowed us to introduce the nondegenerate contravariant tensor of transformation (Jacobian) C, which connects a diagonal tensor of n[th] order $Z_D(\omega^2)$ of a primitive circuit with an impedance tensor of n-th order of an orthogonal pure-loop circuit

$$Z^{(n)}(\omega^2) = C^T Z_D(\omega^2) C$$

and the dual covariant tensor A, which connects the conductance tensor of a primitive circuit with the tensor of an orthogonal pure-node $Y_D(\omega^2)$ circuit

$$Y^{(n)}(\omega^2) = A^T Y_D(\omega^2) A$$

Also, it has been shown in [4,5] that tensors C and A are related as

$$C^T = A^{-1}.$$

If the matrices A and C are divided into blocks in accordance with the division of circuit variables into k loop (contravariant) and n-k node variables, then it turns out that the matrices of the tensors C and A have the following block structures:

$$C = \left|C_k C_{n-k}\right| A = \left|A_k A_{n-k}\right|$$

WhereCk and An-k coincide with the loop and structural matrices of the circuit (Milnikov & Prangishvili 2014).

Proposition 5.

The matrices $Z^{(n)}(\omega^2)$ and $Y^{(n)}(\omega^2)$ are reciprocal, i.e. $Y^{(n)}(\omega^2)$ is the resolvent for, and vice versa.

Indeed, taking into account $Z_D(\lambda) = (Y_D(\lambda))^{-1}$ and also equality (8) we can write

$$\left(Y^{(n)}(\lambda)\right)^{-1} = \left(A^T Y_D(\lambda) A\right)^{-1} = A^{-1} Y_D^{-1}(\lambda)(A^T)^{-1} = C^T Z_D(\lambda) C$$

Analogously, in the dual case we obtain

$$\left(Z^{(n)}(\lambda)\right)^{-1} = \left(C^T Z_D(\lambda) C\right)^{-1} = A^T Y_D(\lambda) A = Y^{(n)}(\lambda)$$

Results

Above we have defined the process of imposing constraints as a successive cutting of the last n-k “fictitious” loops in an orthogonal pure-loop circuit (or as a shorting of the first k “fictitious” node pairs). As a result, we obtain a number of intermediate problems on eigenvalues for operators $Z^{(n-i)}$ (i= 1, ,…, n-k). For i=n-k we have the initial problem for the operator $Z^{(k)}$ (in the dual case we have a number of intermediate problems for operators Y^{n-i} (i= 1, ,…, k)).

The cutting of the j-th loop (j = n; n-1,…, k) is obviously written as an equation (4) i^j =0, whereij is the loop current of the j-th loop. If as a basis operator we take Z^n, then by Proposition 5 its resolvent is Y^n, and for an arbitrary LC-circuit we can write the Weinstein function as follows:

$$W(\lambda) = \left|(Y^{(n)}(\lambda) I^{(i)}, I^{(j)})\right|$$

$$i, j = n, n-1, \dots, k+1$$

Having performed all multiplication operations in (5.23), we obtain the determinant of the matrix of (n-k) order, lying

Journal of Technical Science and Technologies; ISSN 2298-0032; Volume 3, Issue 2, 2014

at the intersection of the last n-k rows and columns of the matrix Y^n , i.e. in the lower right corner of Y^n . This gives rise to

Proposition 6
The Weinstein function for the LC-circuit described by the loop matrix $Z^{(k)}$ is the determinant of a lower right submatrix of order n-k.The dual statement is also valid.

Proposition 7
The Weinstein function for the LC-circuit described by the node matrix $Y^{(n-k)}$ is the determinant of an upper submatrix of order k of the resolvent Zn lying at the intersection of the first k rows and columns of the matrix $Z^{(n)}$.

One can easily establish a relation between these submatrices.

Lemma 1. A lower right submatrix of order (n-k) of the resolventY(n) is a node conductance matrix $Y^{(n-k)}$ of the k-loop circuit.

$$Y^{(n)}(\lambda)=\left\|\begin{matrix}A_k^T\\A_{n-k}^T\end{matrix}\right\|\cdot\|Y_D(\lambda)\|\cdot\|A_kA_{n-k}\|=$$
$$=\left\|\begin{matrix}A_k^TY_d(\lambda)A_k & A_k^TY_d(\lambda)A_{n-k}\\A_{n-k}^TY_d(\lambda)A_k & A_{n-k}^TY_d(\lambda)A_{n-k}\end{matrix}\right\|.$$

The block located in the right lower corner is the node conductance matrix $Y^{(n-k)}$ by virtue of the fact that A coincides with the structural matrix of the circuit. Q.E.D.

The dual statement is proved analogously.

Lemma 2. An upper left upper submatrix of order k of the resolvent $Z^{(n)}$ is a loop impedance matrix $Z^{(k)}$ of the k-loop circuit.

Propositions 6 and 7 and Lemmas 1 and 2 immediately imply

Proposition8
The determinant of the conductance node matrix of an arbitrary k-loop LC-circuit is the Weinstein function for the loop impedance matrix of this circuit, and vice versa.

Conclusion

The propositions establish a deep relationship of the classical loop current and node potential methods with the operator methods of many-dimensional geometry. Thus $A_{n-k}^TY_d(\lambda)A_{n-k}$ is, on the one hand, the Weinstein's function obtained by imposing n-k constraints on the resolvent $Y^{(n)}(\lambda)$ of the operator $Z^{(n)}(\lambda)$ and, on the other hand, the node conductance matrix of the considered circuit. Conversely, if the circuit is considered in terms of node analysis, then the Weinstein's function $C_k^TZ_d(\lambda)C_k$ obtained by imposing k constraints on the resolvent of $Z^{(n)}(\lambda)$ the operator $Y^{(n)}(\lambda)$ is the loop resistance matrix of the analyzed circuit. The all results from a base for developing of full solution and a new algorithm of multi loop electrical circuits' eigenvalues (resonance frequencies) problem.

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